What Makes an Effective Mathematics Lesson?

Developing and Reflecting on Subject Knowledge and Pedagogy in Secondary Mathematics

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In a classroom, mathematics teachers employ many different techniques and theories in the execution of an effective lesson. Covering a range of literature, we analyse communication, questioning, reasoning, and challenge in a mathematics classroom and its effects on learning. Then, I plan and critically analyse a lesson, making use of communication and questioning throughout, comparing my results to the aforementioned literature. Leading on to a reflection of the lesson, I summarise my findings comparing them to previous studies.

Review of Literature

Communication is a key point often mentioned when discussing effective teaching in the classroom. This communication is advocated in two forms; communication between the teacher and the students, and communication between the students and their peers. The Cockroft Report (Cockcroft et al., 1982) emphasised the importance of both these interactions. Moving away from the traditional 'teaching by dictation' often seen at the time, it argued that dialogue and the ability to argue mathematically are important skills needed when learning mathematics, and the introduction of such interactions within a classroom should be encouraged to strengthen a student's understanding.

Davis (1984) develops this idea further, explaining the benefits of 'task-based' interviews conducted by teachers as a way to gauge current understanding. Asking students questions throughout a task and allowing them to explain and justify specific approaches used gives both students an opportunity to reason mathematically, while also allowing the teacher to assess their current knowledge of the topic. This approach links with Rosenshine and his ten 'Principles of Instruction', the sixth of which being to 'check for student understanding'. "The more-effective teachers frequently check to see if all the students are learning the new material." (Rosenshine, 2010, p.18) This lets the teacher gain feedback as to how the material is being received by the students while also preventing misconceptions being developing by spotting early warning signs. Research furthers this point, with Good and Grouws (1979) finding that frequent and effective questioning in the classroom led to higher scores on mathematical tests.

Questioning can be used effectively within 'dialogic teaching', an approach that utilises communication as a focal point, with dialogue being used to progress knowledge. Bakker et al. found that there are many different ways to teach dialogically, but all "stress on the importance of teaching *for* dialogue as well as teaching *through* dialogue" (Bakker et al., 2015, p.1048). This style actively moves away from 'teaching by dictation', encouraging students to ask and answer open ended questions, engaging themselves in dialogic inquiry. One of the most effective ways to check comprehension is through this dialogical type of questioning. "For whereas questions are usually asked in order to find out the answer, teachers and teacherly adults ask questions to which they already know the answer to find out of [sic] children know the answer."

(Mason, 2000, p.104) Asking open questions such as "what have you understood?" rather than closed questions such as "have you understood?" (Sherrington, 2019, p.33) allows the sharing of knowledge, providing a platform to verbally explain understanding and the different ways of thinking about a problem. This is consistent with the findings by the Office for Standards in Education, Children's Services and Skills (OFSTED): "The best questioning probed pupils' knowledge and understanding, with follow-up questions that helped pupils to explain their thinking in depth and refine initial ideas." (OFSTED, 2012, p.35)

This 'higher-order' questioning is supported by the National Centre for Excellence in the Teaching of Mathematics (NCETM), with encouragement of questions being used to promote discussion and "reasoning rather than 'answer getting'" (NCETM, 2008). The rehearsal of, and the elaboration of, knowledge while answering these 'higher-order' questions strengthens connections and relationships between concepts for the students, and helps to improve their long-term retention. Questions asked have to be meaningful though, and used to drive forward the knowledge in students. Meaningless questions do little to support understanding and can have a negative impact on progress, potentially allowing misconceptions to grow. (Jones, 2019)

In support of questioning, Rosenshine argues with his third principle that "[t]he teacher's questions and student discussion are a major way of providing [the] necessary practice [of new material]."(Rosenshine, 2010, p.12) In addition to the checking of knowledge by the teacher, it also provides the students with an opportunity to discuss their working with peers and engage in mathematical conversation, supporting a dialogic approach to teaching. Rosenshine promoted two methods of effective communication between peers; summarising and then sharing the main ideas of a topic with a partner, and telling the answer to a neighbour. Both of these bear a similarity to tasks conducted by Griffin, where he noted that after asking pupils "to explain to the rest of the group what you did" or "explain to a partner where you have become stuck and why you can't move on", the lesson seemingly had more energy as if it had started anew. (Griffin, 1989, p.13) This peer learning has been found to be effective by Topping, who discovered that both peer tutoring and cooperative learning "yield significant gains in academic achievement." (Topping, 2005, p.635)

The implementation of such peer learning can be achieved with relative ease in the classroom through the use of questioning and paired activities, without the need for a big shift in teaching style. A four-phase discussion strategy called "Listen-Think-Pair-Share" (also known as 'Think-Pair-Share') is one such way that mathematical discussion can be generated with peers. "With this strategy, students are taught to listen to the question, think about the question, to discuss the question in pairs, and finally . . . share with the total group."(Lyman Jr., 1981, p.110) Bradbie et al. (1979) found that the use of this strategy not only improved the quality of responses from students when compared with standard classroom questioning, but could also be associated with better recall of the topic. Further to this, Reinhart (2000) found an increase in student confidence and engagement, with 'Think-Pair-Share' being shown to be an effective strategy in improving mathematical discussions.

Standing in contrast to this approach, directed instruction, whereby the teacher provides explicit instructions to the students, provides the teacher with greater opportunities to question and guide student thought, but removes much of the communication between peers. Nevertheless, this approach has been found to be an effective strategy in teaching (Stockard, 2015) and is often used in conjunction with guided practice and independent practice (commonly known as 'I do, We do, You do')(Lombardi, 2019). Kewley found that this approach had a greater impact on student learning during certain problem solving tasks when compared with peer learning, but urges caution when solely implementing one of these strategies. "The fact that both [peer learning and directed instruction] promote learning is supported by scientific evidence. However, . . . [r]ather than suggesting that one type of methodology should be exclusively employed in the classroom, the outcomes of this study emphasize the importance of having students experience a broad range of instructional strategies" (Kewley, 1998, p.31)

Aside from teaching strategies, asking the right type of questions can also encourage mathematical reasoning and thinking. OFSTED (2012) noted that a considerable discussion was generated after a question on multiples was asked in a Y4 class. This idea is also reflected in the NCETM's 'Five big ideas in teaching for mastery', with 'mathematical thinking' being one of these big ideas (NCETM, n.d.). The concept of reasoning and the development of arguments is vital in mathematics, with the one of the three overarching aims of the National Curriculum to be able to "reason mathematically." (Dept. for Education, 2014)

Mathematical thinking and reasoning has not always been at the focus of teaching mathematics, however. Stimulus-Response theory, the learning by means of establishing connections, developed by Thorndike and Skinner (Harris and Spooner, 2000), was a prominent theory throughout the 1900s. Thorndike (1921) developed this into the theory of Drill and Practice with respect to the learning of mathematics, with the doing of repeated exercises providing students with the practice required to learn and answer mathematical questions. This approach, however, fails to make the necessary distinction between doing and understanding, with Davis encouraging problem solving and not just the memorisation of "meaningless prescribed algorithms." (Davis, 1984, p.90) This difference of 'working on' rather than 'working through' problems is explained by Mason (2000), as when a student is allowed to work out a solution through the exploration of a problem, they create their own connections between ideas and formulae, rather than the memorisation and repeated act of substituting values into an equation.

The justification behind the answer underlies this key point in the development of mathematical reasoning, with Skemp (1978) warning against the teaching and learning of 'rules without reason'. He describes two types of understanding in mathematics; 'relational understanding', a deeper mathematical knowledge of concepts and a comprehension of how topics may link together, and 'instrumental understanding', a shallower understanding that is often associated with the sole ability to follow a set of steps. Without deeper questioning by the teacher, the type of understanding a learner is developing is hard to distinguish as both can lead to high scores on tests and effective 'closed question' answering. Skemp argues that teaching to both types of understanding has its benefits, and in a classroom, one has to balance the ease and time efficient teaching of instrumental understanding with the teaching of the more adaptable and sturdier foundations of relational understanding.

However, the Cockroft Report warns against this 'over-simplified' approach when considering understanding, stating that "the distinction between relational and instrumental understanding can never be clear cut . . . but the distinction can be a helpful starting point for discussion of the nature of understanding." (Cockcroft et al., 1982, p.68) The report argues that understanding cannot be observed directly by the teacher, and therefore advocates the use of practical work or problem-solving activities in a classroom, with challenge being needed to compel students to think more deeply about a topic.

Halmos agrees with the use of challenge as an approach to teaching mathematics, driving students to think about methods used rather than just applying algorithms. "Challenge is the best teaching tool there is, for arithmetic as well as for functional analysis, for high-school algebra as well as graduate-school topology." (Halmos, 1985, p.271) Wigley (1992) also expresses the need for challenge as a model for classroom teaching. He writes that the 'challenging model' for the classroom encourages exploration of mathematics while also supporting dialogue between the teacher and students, emphasising the point that this 'challenging model' can easily be implemented into classrooms without much burden on the teacher. Further, Wigley believes that any student of any ability can be challenged, so long as the problems being presented are "pitched at the right level". These ideas link with Bjork's (1994) view that students need to be set work at 'desirable difficulties'. Arguing against the misconception that repeated exposure of material leads to learning, he argues that the setting of varied and challenging work leads to a greater understanding for the students. It is therefore the teacher's role to facilitate challenge during the lesson by providing the opportunity for, as well as actively encouraging, students to attempt challenge questions.

Nevertheless, Wigley urges care when deciding the right amount of challenge for students, with this concern also being raised by Mason (2002). He agrees that challenge is needed in the classroom to further student understanding, but it cannot be too "insufficient or excessive". A problem too easy may not be considered a challenge, disengaging some students and doing little to further knowledge outside of the standard practice question. A problem too hard however may discourage some students from attempting the task to begin with, especially if a foothold cannot be found.

To conclude, communication is key to promoting mathematical reasoning in a classroom, with the most effective teachers using a dialogic approach that encourages communication through peer discussions, such as 'Think-Pair-Share', or teacher led questioning. Questioning should not only be used as a tool to progress knowledge in the students however, but also as a way to assess current understanding and strengthen connections to previous learning. Building this relational understanding is enforced through the use of challenge, with problems being used to strengthen these connections between different mathematical topics.

Nevertheless, caution should be given when implementing such approaches. The overuse of questioning can lead to redundant questioning, which in turn ruins the flow of student thought and can have a negative impact on learning. Likewise, a balance must also be found while implementing the use of challenge, setting work at a desirable difficulty as to not be too "insufficient or excessive". As a consequence of poorly thought out challenges, either extreme can result in little to no progress being made by the pupils.

Implementation

Communication and questioning with the aim of reasoning is therefore important in a classroom, and so I designed a lesson to implement and use these practices. The target class for this study was a low ability Year 8 group consisting of 20 pupils, 12 male and 8 female. This class was chosen due to their performance on a Prior Learning Check (PLC), with many of them showing that they held misconceptions about column multiplication, a topic taught last year. The teaching and review of this topic has not been effective in the past, with students unable to grasp the underlying concepts behind multiplication.

I had noted that similar lessons taught by other teachers employed 'Drill and Practice' in regards to multiplication, with little opportunity for discussion particularly amongst lower sets. In discussions with other teachers at the school, many either saw reasoning with regards to multiplication as unnecessary or were unable to incorporate it into lessons due to a lack of time. This contradicts Mason (2000) as it does not allow students to work on problems and create their own connections with previous knowledge, and no matter a student's ability, reasoning can and should be used as a tool to further understanding in a given topic. This was therefore the purpose of the planned lesson, to use questioning and communication to review knowledge while building their understanding through reasoning, helping the students gain a strong foundation to a fundamental concept in mathematics and allowing them to progress confidently on to later topics. Communication was implemented in two forms; dialogue with the teacher through questioning, explaining answers and the reasoning behind certain steps, and dialogue with peers, paired discussions answering questions and exploring the reason behind those answers.

The lesson started with three starter questions on the board for when the pupils arrived, covering previously taught topics of addition and subtraction, along with the ordering of numbers. Rosenshine (2010) promotes this review of previous knowledge, daily, weekly, and monthly.

Along with the benefit of reviewing topics taught over the last two weeks, the questions also set up thought of place value. Using 'cold calling', a technique endorsed by Lemov (2015), I targeted students that struggled with these topics on the PLC, asking them why they positioned the numbers in columns as they did. Stacking of the numbers is an act that many mathematicians consider trivial, and so the reasoning behind this step is important in setting the foundations for the rest of the lesson. It also shows the students an arguably obvious link between multiplication, and addition and place value. Initial responses from students showed a lack of understanding outside the use of 'rules without reason', but through selected questioning and comparison of digits, asking the students what each digit represented, the class could recognise and explain why the numbers are stacked as so. The use of questioning guided them towards building their own connections with previous knowledge, enforced through the use of review, with 'why' answers not just being granted to them.

Leading on to the next activity, pupils were provided with three questions that required multiplication by factors of 10 and 100. 'Think-Pair-Share' was used to generate discussion around the questions and the process used when working towards an answer. The students were given four minutes to use their previous knowledge to calculate answers and to think about steps taken to reach each answer. Then, they were provided with six minutes to discuss with a partner their mathematical process to achieve each answer. A teacher led discussion followed, with each pair being called upon to share their thoughts and process when tackling the questions. This activity encouraged peer learning between the students, along with providing me the opportunity to analyse comprehension of the class.

The focus of this exercise was not to answer the three questions, but to get the students thinking metacognitively about each step taken while answering the questions and as to why each step worked. The initial time for solo work was provided so that each student could analyse their own initial thoughts and knowledge when approaching the problem before sharing this with a partner. Although unable to 'eavesdrop' on each pair due to current COVID-19 health restrictions, the result of this exercise was very positive. While questioning the process behind answers, every pair could explain why the digits 'shifted' along the place values and how multiplication in two separate steps could be employed, e.g. $4 \times 20 = 4 \times 2 \times 10$. Explaining their deeper thought process, the students seemed engaged and excited to share their own way of thinking, agreeing with Griffin's (1989) observation that the lesson seemingly having more energy after paired discussion.

Moving on to column multiplication of a two digit number by a single digit number, the 'I do, We do, You do' technique was implemented. I reviewing the steps needed to calculate an answer using the column method, provided without discussion of reason as to why they needed to be taken. For the following two questions, I cold called students to explain the steps they would need to take, targeted beforehand based on their PLC. To the first problem, a student said I needed to put a zero in the last column when multiplying a number in the tens column, a step which they could not explain the reasoning behind. Hoping to guide the students to the answer, I asked them to observe each of the digits and think about what they represented. Their initial reaction was frustration, with the initial student commenting: "That's just what you do."

Unplanned, I asked the students to discuss in pairs why the zero was needed. I did this to provide more discussion around a fundamental concept and to build on the need for mathematical communication with others while learning. When asked again after their discussion, the student understood the meaning behind the zero's placement and could explain the link with the previous task, providing the student with an important connection to previous work and understanding. When asked during later lessons, the student showed that they had retained this information and the reasoning behind the zero's placement.

After independent practice, we were unfortunately unable to finish the lesson with the plenary activity due to timings, but rather we started the next lesson with this activity. The exercise involved spotting and correcting a mistake shown in working on the board, and it was used as a tool primarily by me to gauge understanding, but also by the students as a review of the previous lesson's learning. 'Think-Pair-Share' was again implemented to give the students an opportunity to identify the mistake before discussing with a partner. The error was designed to challenge understanding of column multiplication and the pupil's knowledge of the relationship between its steps and place value.

After a discussion, each pair held up a whiteboard, with all but one pair correctly identifying the error alongside a corrected calculation. When asked, a student gave the reason that two zeros had been added in the calculation rather than one. The student clearly showed instrumental understanding of the problem, able to identify and correct the error. Further, following with a question as to why this was incorrect, they explained that the calculation was multiplying by a multiple of ten rather than a multiple of a hundred. This demonstrated that the student had built up the relational understanding required to make the connection between place value and its effects on multiplication, whether it through their own previous knowledge or with the assistance of a peer.

At the end of term, the students sat a test covering all the topics covered so far in lessons. Analysis of the results indicate that the students that struggled with multiplication in the PLC had made progress and could now answer questions requiring the use of column multiplication. I believe this is due to the use of reasoning in the lesson, moving away from simply "answer getting" (NCETM, 2008). Comparison with previous results on the topic disproves the use of Thorndike's (1921) Drill and Practice theory, with students performing better after using reasoning within the lesson. Following the findings made by Mason (2000), the students created their own connections with a previous concept and could then apply that to problems in the future. The creation of these connections was due in part to effective, 'higher-order' questioning, with students being guided towards the relevant concepts to focus on. (NCETM, 2008)

However, during the lesson I found myself asking redundant question, contrary to the advice given by Jones (2019). Although I became aware of this redundant questioning during the lesson, it broke the flow of thought from the students and so one must be more aware of this while conducting questioning. Nevertheless, the use of Davis' (1984) task-based interviews, with students justifying their approach and answers, helped them to gain the relational understanding required for the topic. With planning, the problems proposed guided the students through each step, making the process of 'interviewing' students throughout a trivial exercise. But the preparation of such problems can take time, with Skemp (1978) noting that this time is not always available to the teacher. Implementing this strategy in full for every lesson would have a great impact on students learning and retention, but the time required to prepare such meaningful exercises may not always be available to me in the future.

Even so, retention can be complemented by the use of peer learning, specifically 'Think-Pair-Share', with results found in the lesson being concurrent with those of Bradbie et al. (1979). When asked during the next lesson, the students stated that they enjoyed the use of peer learning and being able to speak to a partner mathematically. One pupil remarked after class that they felt less pressure about getting the correct answer and felt more comfortable talking to a peer rather than to the teacher. Lyman Jr. (1981) also mentions this in their findings, with 'Think-Pair-Share' being effective in raising engagement in the classroom and increasing student confidence. In the future, this technique is one that I shall be using more often as it led to a noticeable increase in engagement and enthusiasm for the lesson from the class as a whole. Implementation of such a technique was done with relative ease and with little extra planning required, agreeing with findings by Topping (2005).

During 'Think-Pair-Share' I would have liked to be sat with students and listened to their conversations, checking that each pair was on task and making progress with the problem. This would have also afforded me the opportunity to talk with each student, encouraging and promoting more communication with the teacher. The NCETM (2008) encourages this approach, re-enforcing the communication generated through questioning. However, due to the current COVID-19 health restrictions, student-teacher interaction was limited, putting a greater focus on the use of questioning in the lesson. I feel this impacted my ability to check student understanding, limiting the feedback that Rosenshine (2010) advocates. Once restrictions have eased, I shall dedicate more time to talk to students directly and spot misconceptions developing.

Moving forward, I would like to apply these strategies to higher ability groups, along with classes in Key Stage 4. From this experience, the use of effective questioning pushed the students to think about their understanding of a fundamental mathematical topic. For higher years and abilities, having a greater, relational understanding of previous concepts can help explore non-trivial problems that do not follow the standards steps of calculation. Arguably this is what mathematics teaching is about, teaching problem solving and mathematical reasoning; ideas strongly embedded in the NCETM's 'Five big ideas in teaching mastery.' (NCETM, n.d.) The National Curriculum has developed overtime to reflect this view also, with the Cockroft Report suggesting that communication and problem solving in the classroom leads to effective learning, preparing them for employment with effective 'real-world' skills (Cockcroft et al., 1982). Encouraging peer communication in Key Stage 4 would give students the opportunity to reason verbally and engage in mathematical conversations, a skill that can be carried with them into the future.

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