A Proof of the Irrationality of π

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The following proof was set by Mary Cartwright as an exercise in a Cambridge University exam in 1945. Although its origins are unknown, it is a simplification an earlier proof by Charles Hermite. Intermediate steps have been provided as to make each step clearer for the reader.

Theorem. π *is irrational.*

Proof. Consider the integrals

$$I_n(x) = \int_{-1}^{1} (1 - z^2)^n \cos(xz) \, \mathrm{d}z$$

where $n \in \mathbb{N}_0$. Integrating by parts,

$$I_n(x) = \frac{2n}{x} \int_{-1}^{1} z(1 - z^2)^{n-1} \sin(xz) \, \mathrm{d}z.$$

Integrating by parts again,

$$I_n(x) = \frac{2n}{x^2} \int_{-1}^{1} (1 - z^2)^{n-1} \cos(xz) - 2(n-1)z^2(1 - z^2)^{n-2} \cos(xz) \, \mathrm{d}z$$

$$\implies \quad x^2 I_n(x) = 2nI_{n-1}(x) - 4n(n-1) \int_{-1}^{1} z^2(1 - z^2)^{n-2} \cos(xz) \, \mathrm{d}z.$$

We know that

$$(1 - z^2)^{n-1} = (1 - z^2)^{n-2} - z^2(1 - z^2)^{n-2}$$

$$\implies \quad z^2(1 - z^2)^{n-2} = (1 - z^2)^{n-2} - (1 - z^2)^{n-1},$$

so

$$\begin{aligned} x^2 I_n(x) &= 2nI_{n-1}(x) - 4n(n-1)\int_{-1}^{1}(1-z^2)^{n-2}\cos(xz)\ \mathrm{d}z \\ &\qquad + 4n(n-1)\int_{-1}^{1}(1-z^2)^{n-1}\cos(xz)\ \mathrm{d}z \\ &= 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x). \end{aligned}$$

Thus we have the recurrence relation

$$x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x)$$

for $n \geq 2$. Letting

$$J_n(x) = x^{2n+1} I_n(x), \tag{1}$$

we get

$$\begin{aligned} & x^{-2n+1}J_n(x) = 2n(2n-1)x^{-2n+1}J_{n-1}(x) - 4n(n-1)x^{-2n+3}J_{n-2}(x) \\ \Longrightarrow \quad & J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x). \end{aligned}$$

We now see that

$$\begin{aligned} J_0(x) &= x\int_{-1}^{1}(1-z^2)^0\cos(xz)\ \mathrm{d}z \\ &= 2\sin(x) \end{aligned}$$

and

$$\begin{aligned} J_1(x) &= x^3\int_{-1}^{1}(1-z^2)^1\cos(xz)\ \mathrm{d}z \\ &= x^3\int_{-1}^{1}\cos(xz)\ \mathrm{d}z - x^3\int_{-1}^{1}z^2\cos(xz)\ \mathrm{d}z \\ &= 2x^2\sin(x) - x^3\int_{-1}^{1}z^2\cos(xz)\ \mathrm{d}z. \end{aligned}$$

Integrating by parts twice,

$$\begin{aligned} J_1(x) &= 2x^2\sin(x) - x^3\left[\frac{2}{x}\sin(x) - \frac{2}{x}\int_{-1}^{1}z\sin(xz)\ \mathrm{d}z\right] \\ &= 2x^2\int_{-1}^{1}z\sin(xz)\ \mathrm{d}z \\ &= 2x^2\left[-\frac{2}{x}\cos(x) + \frac{1}{x}\int_{-1}^{1}\cos(xz)\ \mathrm{d}z\right] \\ &= -4x\cos(x) + 4\sin(x). \end{aligned}$$

As

$$\begin{aligned} J_n(x) &= 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x) \\ &= n\left(2(2n-1)J_{n-1}(x) - 4(n-1)x^2 J_{n-2}(x)\right) \end{aligned}$$

and

$$\begin{aligned} J_0(x) &= 2\sin(x) \\ J_1(x) &= -4x\cos(x) + 4\sin(x), \end{aligned}$$

for all $n \in \mathbb{N}_0$,

$$J_n(x) = n!\,(P_n(x)\sin(x) + Q_n(x)\cos(x)) \tag{2}$$

where $P_n(x)$ and $Q_n(x)$ are polynomials of degree $\leq n$ with integer coefficients dependent on n.
Now we take $x = \frac{\pi}{2}$ and assume for contradiction that $\frac{\pi}{2} = \frac{a}{b}$ where $a, b \in \mathbb{N}$ and are coprime, i.e. assume π is rational. By (1) and (2),

$$\begin{aligned} & \left(\frac{\pi}{2}\right)^{2n+1} I_n\left(\frac{\pi}{2}\right) = n!\left(P_n\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + Q_n\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)\right) \\ = \quad & \frac{a^{2n+1}}{b^{2n+1}} I_n\left(\frac{\pi}{2}\right) = n! P_n\left(\frac{\pi}{2}\right) \\ \implies \quad & \frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) = P_n\left(\frac{\pi}{2}\right) b^{2n+1}. \end{aligned}$$

Now, for all $z \in [-1, 1]$ and for all $n \in \mathbb{N}$,

$$0 \leq \cos\left(\frac{\pi}{2}z\right) \leq 1$$

and

$$0 \leq (1-z^2)^n \leq 1,$$

thus, as the function is continuous and not constant,

$$0 < (1-z^2)^n \cos\left(\frac{\pi}{2}z\right) < 1.$$

Hence, as the interval $[-1, 1]$ has length 2,

$$0 < I_n\left(\frac{\pi}{2}\right) < 2.$$

Also,

$$\frac{a^{2n+1}}{n!} \to 0$$

as $n \to \infty$, hence, for sufficiently large n,

$$0 < \frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) < 1.$$

Contradiction, as

$$\frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) = P_n\left(\frac{\pi}{2}\right) b^{2n+1} \in \mathbb{Z},$$

hence $\frac{\pi}{2} \notin \mathbb{Q}$ and thus π is irrational. $\square$