MATH319

P16: PID Control of Wind Turbines for Clean Energy

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The blades of a wind turbine rotate that in-turn spins a turbine inside, generating electricity. With varying wind speeds, the blade pitch can be slightly adjusted to regulate the turbine so that it is generating power at an optimum rate. This can be modelled as a linear system, and we explore the stability of this system and it's components with system gain K, torque of the generator τ, damping ratio ζ, and natural frequency of the system ω.

We shall first begin by finding the poles of the transfer function

$$G(s) = \frac{1}{\tau s + 1} \frac{K\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

for constants $\omega, \tau, \zeta > 0$ and $K \in \mathbb{R}$. We can clearly see by the denominator of $G(s)$ that $s = -1/\tau$ is a pole for this transfer function. The other poles are calculated by letting

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

and then completing the square.

$$\begin{aligned} & s^2 + 2\zeta\omega s + \omega^2 = 0 \\ = \quad & (s + \zeta\omega)^2 - \zeta^2\omega^2 + \omega^2 = 0 \\ \implies \quad & (s + \zeta\omega)^2 = \omega^2(\zeta^2 - 1) \\ \implies \quad & s = -\zeta\omega \pm \omega\sqrt{\zeta^2 - 1} \\ & = \omega\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right), \end{aligned}$$

giving us three poles for $G(s)$, namely

$$\begin{aligned} s_1 &= -1/\tau \\ s_2 &= \omega(-\zeta + \sqrt{\zeta^2 - 1}) \\ s_3 &= \omega(-\zeta - \sqrt{\zeta^2 - 1}). \end{aligned}$$

As $\tau > 0$, we can see that s_1 is negative and hence lies in the complex left half-plane (LHP). As $\zeta, \omega > 0$ also, for $0 < \zeta \leq 1$

$$\zeta^2 - 1 < 0 \implies \sqrt{\zeta^2 - 1} \in \mathbb{C},$$

and for $\zeta > 1$

$$\sqrt{\zeta^2 - 1} < \zeta \implies -\zeta \pm \sqrt{\zeta^2 - 1} < 0,$$

so s_2 and s_3 both lie in the LHP meaning that $G(s)$ is a stable function.
We now introduce the *PID* controller

$$V(s) = \frac{as^2 + bs + c}{s}$$

for some a, b, c. To obtain the transfer function

$$L(s) = \frac{1}{1 + V(s)G(s),}$$

which is the systems sensitivity to disturbances, and the related transfer functions

$$\frac{V(s)}{1 + V(s)G(s)}, \quad \frac{G(s)}{1 + V(s)G(s)}, \quad \frac{V(s)G(s)}{1 + V(s)G(s)},$$

we shall define the matrix $\Psi(s)$ such that

$$\Psi(s) := \frac{1}{1+V(s)G(s)} \begin{bmatrix} 1 & G(s) \\ V(s) & V(s)G(s) \end{bmatrix}$$
$$= \begin{bmatrix} \Psi_{1,1}(s) & \Psi_{1,2}(s) \\ \Psi_{2,1}(s) & \Psi_{2,2}(s) \end{bmatrix}.$$

We can see that $\Psi_{1,1}(s) = L(s)$, $\Psi_{1,2}(s) = \frac{G(s)}{1+V(s)G(s)}$, $\Psi_{2,1}(s) = \frac{V(s)}{1+V(s)G(s)}$, and $\Psi_{2,2}(s) = \frac{V(s)G(s)}{1+V(s)G(s)}$, which is the closed loop transfer function.
We start by substituting $P(s)$ and $V(s)$ into $\Psi(s)$,

$$\Psi(s) = \frac{1}{1+V(s)G(s)} \begin{bmatrix} 1 & G(s) \\ V(s) & V(s)G(s) \end{bmatrix}$$

$$= \frac{1}{1+V(s)G(s)} \begin{bmatrix} 1 & \frac{1}{\tau s+1}\frac{K\omega^2}{s^2+2\zeta\omega s+\omega^2} \\ \frac{as^2+bs+c}{s} & \frac{as^2+bs+c}{s}\cdot\frac{1}{\tau s+1}\frac{K\omega^2}{s^2+2\zeta\omega s+\omega^2} \end{bmatrix}$$

$$= \frac{1}{1+V(s)G(s)}\cdot\frac{1}{\Lambda(s)} \begin{bmatrix} s(\tau s+1)(s^2+2\zeta\omega s+\omega^2) & K\omega^2 s \\ (as^2+bs+c)(\tau s+1)(s^2+2\zeta\omega s+\omega^2) & k\omega^2(as^2+bs+c) \end{bmatrix} \tag{1.1}$$

with $\Lambda(s) := s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)$.
Now finding $\frac{1}{1+V(s)G(s)}$,

$$1+V(s)G(s) = 1 + \frac{as^2+bs+c}{s}\cdot\frac{1}{\tau s+1}\frac{K\omega^2}{s^2+2\zeta\omega s+\omega^2}$$
$$= \frac{s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)+(as^2+bs+c)(K\omega^2)}{s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)}$$
$$\implies \frac{1}{1+V(s)G(s)} = \frac{s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)}{s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)+(as^2+bs+c)(K\omega^2)}$$
$$= \frac{\Lambda(s)}{\Delta(s)} \tag{1.2}$$

for $\Delta(s) := s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)+(as^2+bs+c)(K\omega^2)$, we notice that by (1.2) and (1.1) we obtain

$$\Psi(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s(\tau s+1)(s^2+2\zeta\omega s+\omega^2) & K\omega^2 s \\ (as^2+bs+c)(\tau s+1)(s^2+2\zeta\omega s+\omega^2) & k\omega^2(as^2+bs+c) \end{bmatrix}$$

with

$$
\begin{aligned}
L(s) = \frac{1}{1+V(s)G(s)} &= \Psi_{1,1}(s) = \frac{s(\tau s+1)(s^2+2\zeta\omega s+\omega^2)}{\Delta(s)},\\
\frac{G(s)}{1+V(s)G(s)} &= \Psi_{1,2}(s) = \frac{K\omega^2 s}{\Delta(s)},\\
\frac{V(s)}{1+V(s)G(s)} &= \Psi_{2,1}(s) = \frac{(as^2+bs+c)(\tau s+1)(s^2+2\zeta\omega s+\omega^2)}{\Delta(s)},\\
\frac{V(s)G(s)}{1+V(s)G(s)} &= \Psi_{2,2}(s) = \frac{K\omega^2(as^2+bs+c)}{\Delta(s)}.
\end{aligned}
$$

Using this, we can see that $L(s)$, $\Psi_{1,2}(s)$, $\Psi_{2,1}(s)$, and $\Psi_{2,2}(s)$ are all stable when the poles of $1/\Delta(s)$ lie in the LHP. Expanding $\Delta(s)$, we get

$$
\Delta(s) = \tau s^4 + (2\tau\zeta\omega+1)s^3 + (\tau\omega^2+2\zeta\omega+aK\omega^2)s^2 + (\omega^2+bK\omega^2)s + cK\omega^2.
$$

We now define a Hurwitz polynomial before introducing the Routh Array needed for the Routh-Hurwitz stability criterion.

Definition (Hurwitz)**.** An n-th order polynomial

$$
p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0
$$

with $a_i \in \mathbb{R}$ for $i \in \mathbb{N}_0$, $a_n > 0$, and $a_0 \neq 0$, is a Hurwitz polynomial if its roots lie in the open left half-plane.

Definition (Routh Array)**.** The Routh Array for a polynomial $p(s)$ is defined as

s^n	a_n	a_{n-2}	a_{n-4}	$\dots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\dots$
s^{n-2}	b_1	b_2	b_3	$\dots$
s^{n-3}	c_1	c_2	c_3	$\dots$
s^{n-4}	d_1	d_2	d_3	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
s^2	e_1	e_2	0	$\dots$
s^1	f_1	0	0	$\dots$
s^0	g_0	0	0	$\dots$

for coefficients

$$
b_i = \frac{a_{n-1}a_{n-2i} - a_n a_{n-(2i+1)}}{a_{n-1}}, \quad c_i = \frac{b_1 a_{n-(2i+1)} - a_{n-1}b_{i+1}}{b_1}, \quad d_i = \frac{c_1 b_{i+1} - b_1 c_{i+1}}{c_1}, \ \dots .
$$

The coefficients for the Routh Array are calculated from the Hurwitz determinants which is left for the interested reader (see Peet [2014]).

Theorem: Routh-Hurwitz. *$p(s)$ is Hurwitz if and only if each element of the first column of the Routh Array is positive.*

We omit the proof which is left for further reading (see Anagnost and Desoer [1991]).

Using $\Delta(s)$, we start by finding the Routh Array and then setting up inequalities based on its stability. So the Routh Array for $\Delta(s)$ is

s^4	τ	$\omega(2\zeta+\tau\omega+aK\omega)$	$cK\omega^2$
s^3	$2\tau\omega\zeta+1$	$\omega^2(1+bK)$	0
s^2	$\omega(2\zeta+\tau\omega+aK\omega)-\dfrac{\tau\omega^2(1+bK)}{2\tau\omega\zeta+1}$	$cK\omega^2$	0
s^1	$\omega^2(1+bK)-\dfrac{cK\omega(2\tau\omega\zeta+1)^2}{(2\tau\omega\zeta+1)(2\zeta+\tau\omega+aK\omega)-\tau\omega(1+bK)}$	0	0
s^0	$cK\omega^2$	0	0

which implies $\Delta(s)$ is stable by Routh-Hurwitz when

$$\begin{aligned}
0 &< 2\tau\omega\zeta+1 \\
0 &< \omega(2\zeta+\tau\omega+aK\omega)-\frac{\tau\omega^2(1+bK)}{2\tau\omega\zeta+1} \\
0 &< \omega^2(1+bK)-\frac{cK\omega(2\tau\omega\zeta+1)^2}{(2\tau\omega\zeta+1)(2\zeta+\tau\omega+aK\omega)-\tau\omega(1+bK)} \\
0 &< cK\omega^2
\end{aligned}$$

for $\omega, \tau, \zeta > 0$ and $K \in \mathbb{R}$.
We will explore Nyquist plots and stability for specific values, using the inequalities above to check stability, so let us first consider when $\omega = 10$, $\tau = 8$, $\zeta = 0.05$, and $K = 5000$. Letting $a = 0.002$, $b = 0.0003$, and $c = 0.001$, we check that

$$\begin{aligned}
9 &= 2\tau\omega\zeta+1 \\
\frac{14209}{9} &= \omega(2\zeta+\tau\omega+aK\omega)-\frac{\tau\omega^2(1+bK)}{2\tau\omega\zeta+1} \\
\frac{41274}{167} &= \omega^2(1+bK)-\frac{cK\omega(2\tau\omega\zeta+1)^2}{(2\tau\omega\zeta+1)(2\zeta+\tau\omega+aK\omega)-\tau\omega(1+bK)} \\
500 &= cK\omega^2
\end{aligned}$$

are all greater than 0. Plotting the Nyquist diagrams for $L(s)$ (Figure 1) and the closed loop transer function $\Psi_{2,2}(s)$ (Figure 2), we see that as neither winds around -1, both are stable (see Appendix A.1.1 for MATLAB code and Appendix A.1.2 for confirmation of stability for $\Psi_{1,2}(s)$ & $\Psi_{2,1}(s)$).

Figure 1: Nyquist plot of $L(s)$

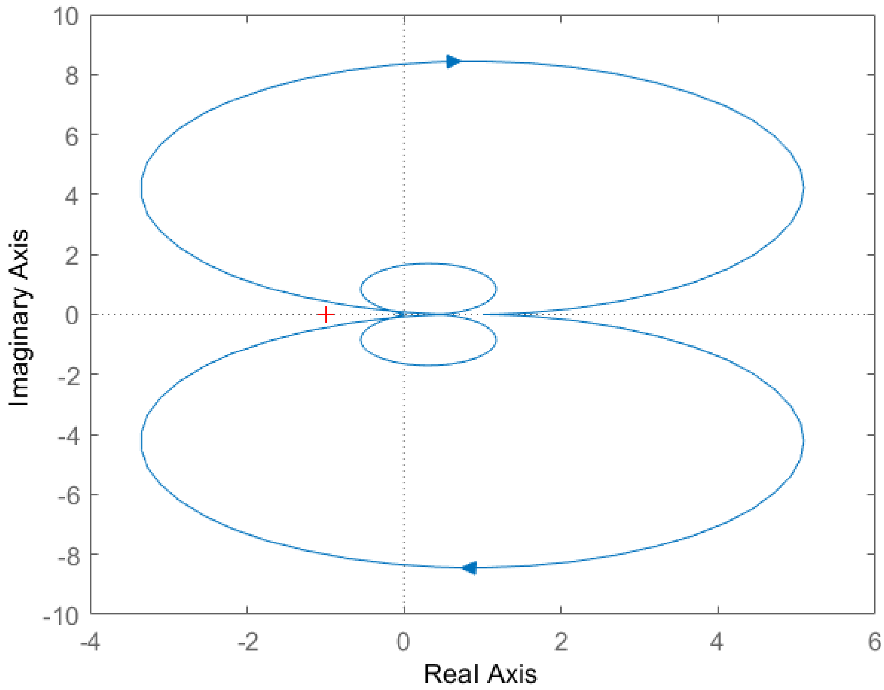
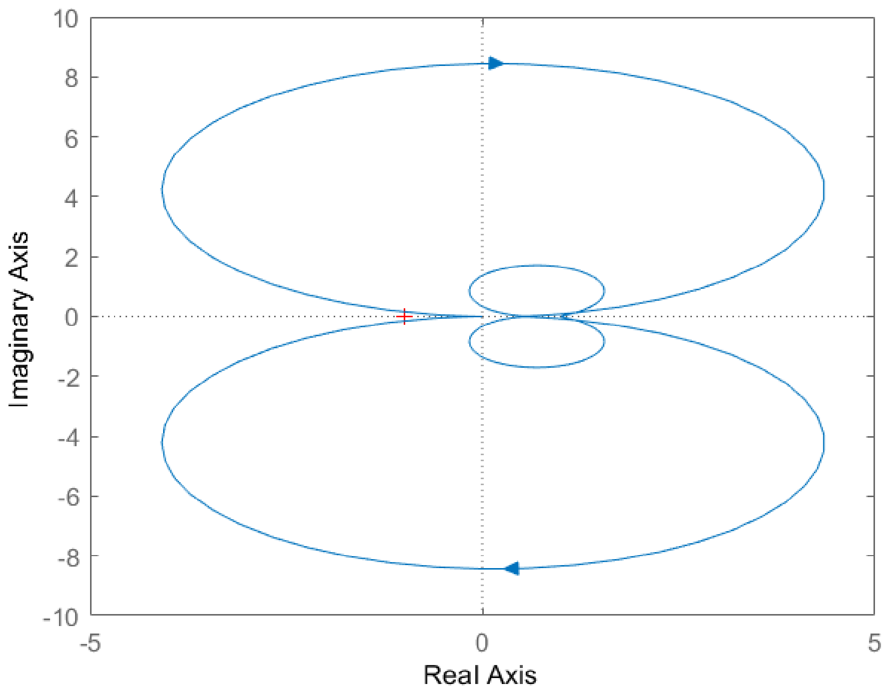


Figure 2: Nyquist plot of $\Psi_{2,2}(s)$



We can also confirm this stability by observing that the poles of $L(s)$ and $\Psi_{2,2}(s)$ lie in the LHP (Figures 3a & 3b, see A.1.3 for code).

(a) Pole-Zero plot for $L(s)$ (b) Pole-Zero plot for $\Psi_{2,2}(s)$

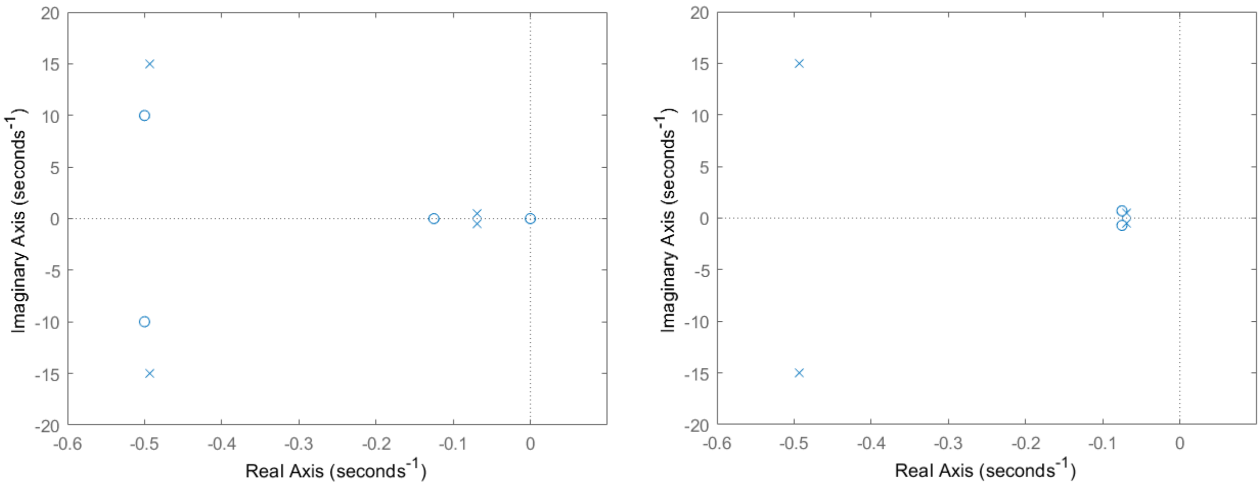


Figure 3: Pole-Zero maps (×-pole, ○-zero)

Now let us consider $\tau, \zeta, \omega, K, b$, and c as above, but let $a = -0.002$. We see that as

$$\omega(2\zeta + \tau\omega + aK\omega) - \frac{\tau\omega^2(1+bK)}{2\tau\omega\zeta + 1} = -\frac{3791}{9} < 0,$$

$L(s)$ and its related transfer functions are not all stable. By plotting the Nyquist plot of $L(s)$ and $\Psi_{2,2}(s)$ (Figures 4 & 5) we see that although $\Psi_{2,2}(s)$ is stable, $L(s)$ wraps around -1 and is therefore not stable.

Figure 4: Nyquist plot of $L(s)$

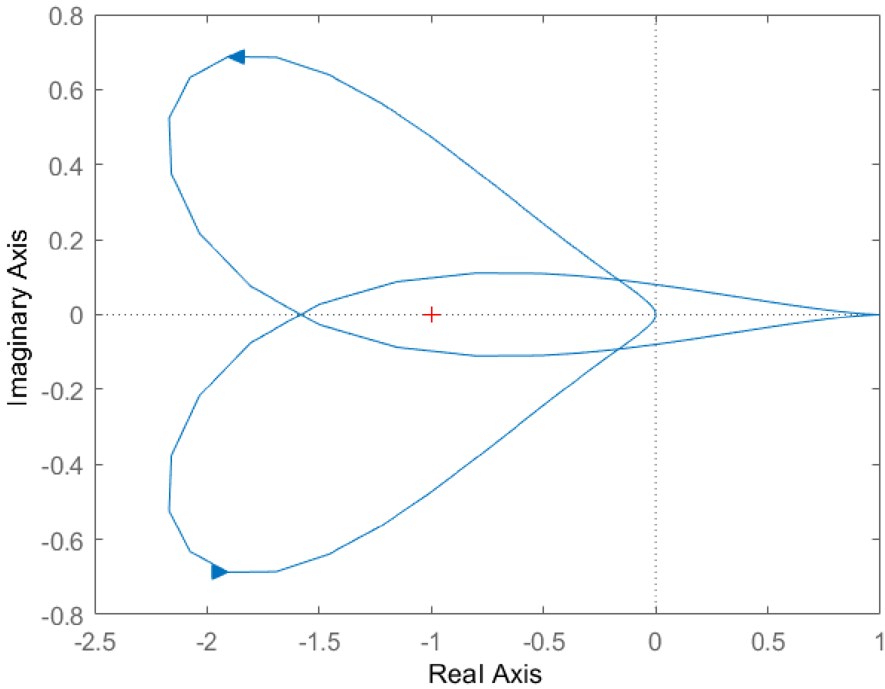
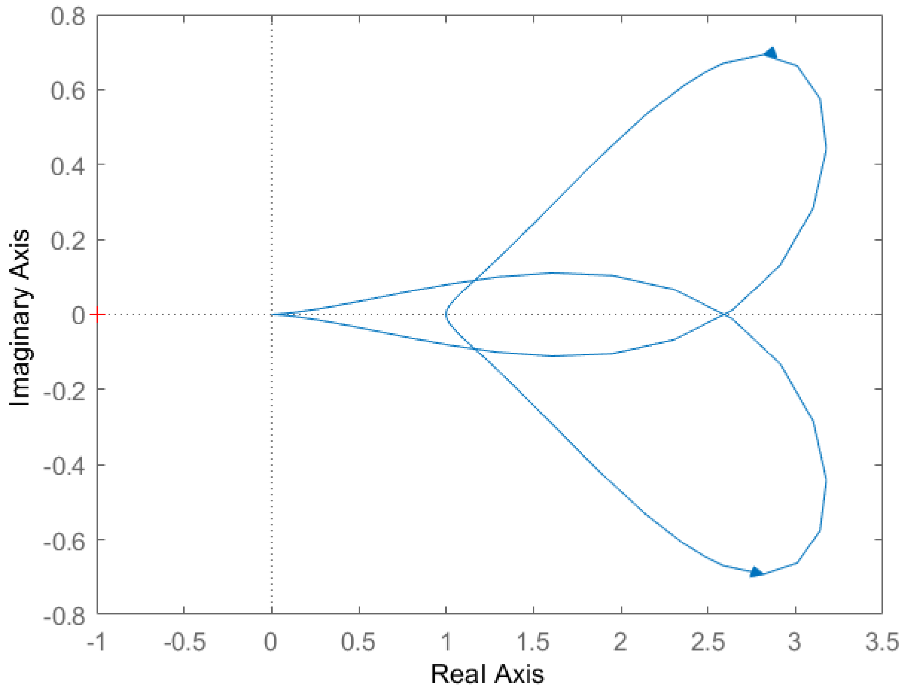


Figure 5: Nyquist plot of $\Psi_{2,2}(s)$



In conclusion, there exists a *PID* controller such that not only the wind turbine as a whole is stable, but also each individual component, such as the generator and pitch motor, is stable. We also observed that changing this *PID* controller slightly can lead to some components becoming unstable, whilst others remain stable.

References

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S. H. Żak. Routh's Stability Criterion. In *ECE 680 Modern Automatic Control*. Purdue University, Jun 2007. West Lafayette, IN. http://www.engr.iupui.edu/~skoskie/ECE680/Routh.pdf.

S. J. P. Chapman. Private Communication, Feb 2019. Dept. of Engineering, University of Cambridge. Discussed the sensitivity and gain due to disturbances in a system.

A MATLAB code

Define variables and transfer functions

```
>> syms a b c t z K w;
>> a=0.002;
>> b=0.0003;
>> c=0.001;
>> K=5000;
>> z=0.05;
>> t=8;
>> w=10;
>> L=tf([t 2*t*z*w+1 t*w^2+2*z*w w^2 0],[t 2*t*z*w+1 t*w^2+2*z*w+a*K*w^2
   w^2+b*K*w^2 c*K*w^2]);

>> P12=tf([K*w^2 0],[t 2*t*z*w+1 t*w^2+2*z*w+a*K*w^2 w^2+b*K*w^2 c*K*w^2]);

>> P21=tf([a*t 2*a*t*z*w+b*t+a a*t*w^2+2*b*t*z*w+c*t+2*a*z*w+b b*t*w^2+2*c*z*w*t
   +a*w^2+2*b*z*w+c c*t*w^2+b*w^2+2*c*z*w c*w^2],[t 2*t*z*w+1 t*w^2+2*z*w+a*K*w^2
   w^2+b*K*w^2 c*K*w^2]);

>> P22=tf([a*K*w^2 b*K*w^2 c*K*w^2],[t 2*t*z*w+1 t*w^2+2*z*w+a*K*w^2 w^2+b*K*w^2
   c*K*w^2]);
```

A.1.1 Figures 1 and 2

```
>> nyquist(L)
>> nyquist(P22)
```

A.1.2 Stability of $\Psi_{1,2}(s)$ and $\Psi_{2,1}(s)$

```
>> isstable(P12)

ans =
logical
1
```

Stability of $\Psi_{1,2}(s)$ is true

```
>> isstable(P21)

ans =
logical
1
```

A.1.3 Figures 3a and 3b

```
>> pzmap(L)
>> xlim([-0.6,0.1])
>> ylim([-20,20])
```

```
>> pzmap(P22)
>> xlim([-0.6,0.1])
>> ylim([-20,20])
```

A.1.4 Figures 4 and 5

```
>> a=-0.002;

>> nyquist(L)
>> nyquist(P22)
```