Mathematics Dissertation

The Hausdorff Dimension of Fractal Sets

JOSEPH ROBERT WEBSTER

Supervisor: Prof. Stephen Power

April 22, 2020

Mathematics & Statistics | Lancaster University

Abstract

Fractal sets often do not hold properties associated with an n-dimensional space. This dissertation explores an alternative to the usual integer dimensions, using the Hausdorff measure $\mathcal{H}^s$ to introduce the Hausdorff dimension $\dim_H$. Using upper and lower bounds of $\mathcal{H}^s$, the Hausdorff dimension of the middle-thirds Cantor set is calculated, which is shown to be non-integer. By introducing the mass distribution principle, the Hausdorff dimension of the von Koch curve and general Cantor sets are calculated. Through potential theory, properties of $\dim_H$ for projected sets are given, with medical applications such as modelling of the brain for analysing Alzheimer's disease being noted.

Contents

1 Introduction

This dissertation focuses on the Hausdorff dimension, exploring different approaches in calculating its value for fractal sets, including the use of the mass distribution principle and projections. Most of the results are based upon the work of Falconer [8], but some theorems and concepts have been expanded upon with more explicit steps guiding the reader (e.g. Lemma 2.28, Section 2.4).

Section 1 starts with the preliminary results needed for later use; these are not always directly related to the following material and thus are positioned here.
Introducing fractals sets in Section 2, we lead into a discussion about measures. A specific measure, the Hausdorff measure, is then introduced and is used to define the Hausdorff dimension of a set. Many of the results here are found in Falconer, but supplementary proofs and theorems (e.g. Lemma 2.14, Proposition 2.19) are provided from a variety of sources, most notably from Bartle [1].
Addressing the problems with calculating this dimension, in Section 3 we discuss mass distributions and, after discussing Borel sets, we use integrals to find the potential and energy of a measure. Again, the results here follows the work of Falconer, with additional examples and detail being added (e.g. Section 3.1).
This leads into Section 4, using the theory of energy to discuss the dimension of projections of sets. We finish with a thought on abstract fractal sets and medical applications of the fractal dimension.

1.1 Preliminary Results

We begin with some preliminary definitions and results that will be used throughout.

Definition 1.1 *(Rudin [21]).* Let X be a set and let τ be a collection of subsets of X. A *topological space* is the ordered pair (X, τ) such that τ has the following properties:

(O1) $X \in \tau$ and $\emptyset \in \tau$.

(O2) If $\{V_i\}$ is an arbitrary collection of sets in τ, then the union $\bigcup_i V_i \in \tau$.

(O3) If $\{V_j\}$ is a finite collection of sets in τ, then the intersection $\bigcap_j V_j \in \tau$.

We call τ a *topology in* X and elements of τ are called the *open sets* in X.

Example 1.1 (Lipschutz [15]). Let $\mathscr{U}$ be the collection of all open sets of real numbers. Then $\mathscr{U}$ is a topology on $\mathbb{R}$, called the *usual topology on* $\mathbb{R}$.
Similarly, the collection $\mathscr{V}$ of all open sets in the plane $\mathbb{R}^2$ is a topology on $\mathbb{R}^2$, called the *usual topology on* $\mathbb{R}^2$.

Example 1.2 (Lipschutz [15]). Consider the discrete set $X = \{a, b, c, d, e\}$ and the following collection of subsets;

$$\begin{aligned}
\tau_1 &= \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\} \\
\tau_2 &= \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\} \\
\tau_3 &= \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}.
\end{aligned}$$

We see that τ_1 satisfies the conditions above and thus is a topology of X.
Considering τ_2 on the other hand, since the union

$$\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin \tau_2,$$

we see that (O2) is not satisfied by τ_2, and thus it is not a topology of X.
Also, τ_3 is not a topology of X as considering the intersection

$$\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \tau_3,$$

we see that (O3) is not satisfied.

Definition 1.2 *(Falconer [8]).* A subset $F \subset \mathbb{R}^n$ is *open* if for all $x \in F$, there exists a ball $B(x, r)$ centred at x with radius $r > 0$ contained in F.

Definition 1.3 *(Falconer [8]).* A subset $F \subset \mathbb{R}^n$ is *closed* if its complement is open. Equivalently, F is closed if for any convergent sequence x_n in F, its limit x in $\mathbb{R}^n$ is also contained in F.

Definition 1.4 *(Falconer [8]).* A finite/countable collection of non-empty subsets $\{U_i\}$ in $\mathbb{R}^n$ is a *cover* of a set $F \subset \mathbb{R}^n$ if

$$F \subset \bigcup_{i=1}^{\infty} U_i.$$

Definition 1.5 *(Falconer [8]).* A subset F of a topological space is *compact* if every open cover $\{U_i\}$ of F, where U_i are open sets, has a finite subcover of F.

Theorem 1.6 *(**Heine-Borel Theorem**, Belton [2, Theorem 2.23]). Suppose $S \subset \mathbb{R}^n$, then S is compact if and only if it is closed and bounded.*

Proof. Omitted, but can be proven using the *Bolzano-Weierstrass theorem.* $\square$

Theorem 1.7 *(**De Morgan's law**, Grabowski [9, Theorem 2.28]). Suppose $A_i \subset \mathbb{R}^n$ for all $i \in I$ where I is some indexing set, then*

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i^c),$$

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i^c).$$

Proof. Omitted as this can easily be proven from the definitions. □

Proposition 1.8 *(Jerrum [12]). The union of an arbitrary number of open sets is open, and the intersection of a finite number of open sets is open.*

Proof. First suppose that $\{A_i\}$ is a collection of open sets indexed by some I. Let

$$x \in \bigcup_{i \in I} A_i,$$

such that x belongs to at least one open A_i. Thus there exists a ball $B(x,r) \subseteq A_i$ centred at x with radius $r > 0$, and hence

$$B(x,r) \subseteq \bigcup_{i \in I} A_i.$$

Since x was chosen arbitrarily, the union $\bigcup_{i \in I} A_i$ is open.

Now suppose that $\{A_i\}$ is a finite collection of open sets. Let

$$x \in \bigcap_{i=1}^{n} A_i,$$

such that $x \in A_i$ for all $i = 1, 2, \ldots, n$. For each open A_i, there exists a ball $B(x, r_i) \subseteq A_i$ centred at x with radius $r_i > 0$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$, so

$$B(x,r) \subseteq B(x,r_i) \subseteq A_i$$

for all i, and thus

$$B(x,r) \subseteq \bigcap_{i=1}^{n} A_i.$$

Since x was chosen arbitrarily, the intersection $\bigcap_{i=1}^{n} A_i$ is open. □

Proposition 1.9 *(Math Online [17]). The union of a finite number of closed sets is closed, and the intersection of an arbitrary number of closed sets is closed.*

Proof. First suppose that $\{A_i\}$ is a finite collection of closed sets, and let

$$S = \bigcup_{i=1}^{n} A_i.$$

By De Morgan's law,

$$S^c = \left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Each A_i is closed, so A_i^c is open, and thus the intersection $\bigcap_{i=1}^n A_i^c$ is open. Hence $(S^c)^c = S$ is closed, therefore the union $\bigcup_{i=1}^n A_i$ is closed.

Now suppose that $\{A_i\}$ is a collection of closed sets indexed by some I, and let

$$T = \bigcap_{i \in I} A_i.$$

By De Morgan's law,

$$T^c = \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

Each A_i is closed, so A_i^c is open, and thus the union $\bigcup_{i \in I} A_i^c$ is open. Hence $(T^c)^c = T$ is closed, therefore the intersection $\bigcap_{i \in I} A_i$ is closed. □

Definition 1.10 *(Falconer [8]).* The *closure* $\overline{F}$ of $F \subset \mathbb{R}^n$ is the smallest closed set containing F.

Definition 1.11 *(Falconer [8]).* Let $X \subset \mathbb{R}$. The set $F \subset X$ is *dense* in X if $\overline{F} = X$.
Equivalently, $F \subset X$ is dense in X if for each $x \in X$ and all $\varepsilon > 0$, there exists $z \in F$ such that $|z - x| < \varepsilon$.

Definition 1.12 *(Falconer [8]).* Suppose that U is a non-empty subset of $\mathbb{R}^n$. The *diameter* of U is

$$|U| := \sup\{|x - y| : \; x, y \in U\}.$$

We let

$$|\emptyset| = 0.$$

Remark. Some sources denote the diameter of a set U as $diam(U)$ or $||U||$.

Proposition 1.13 *(Rudin [20]). Suppose that U is a non-empty subset of $\mathbb{R}^n$ with closure $\overline{U}$, then*

$$|U| = \left|\overline{U}\right|.$$

Proof. Since $U \subset \overline{U}$, it follows that

$$|U| \leq |\overline{U}|.$$

For the reverse inequality, fix $\varepsilon > 0$ and choose $x, y \in \overline{U}$. Clearly U is dense in $\overline{U}$ by the definition of dense, so there exists $x', y' \in U$ such that

$$|x - x'| < \varepsilon \quad \text{and} \quad |y - y'| < \varepsilon.$$

Hence, by the triangle inequality,

$$\begin{aligned} |x-y| &\le |x-x'| + |x'-y'| + |y'-y| \\ &< 2\varepsilon + |x'-y'| \\ &\le 2\varepsilon + |U|. \end{aligned}$$

Thus

$$|\overline{U}| \le 2\varepsilon + |U|,$$

and as ε is arbitrary, the result follows. □

Definition 1.14 *(Falconer [8]).* Suppose $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$. The mapping f satisfies a *Hölder condition for exponent* α if

$$|f(x) - f(y)| \le c|x-y|^\alpha$$

for $x, y \in F$ and constants $\alpha, c > 0$.
The mapping f is said to be *Lipschitz* when $\alpha = 1$.

Definition 1.15 *(Falconer [8]).* Suppose $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$. The mapping f is said to be *bi-Lipschitz* if

$$c_1|x-y| \le |f(x) - f(y)| \le c_2|x-y|$$

for constants $0 < c_1 \le c_2 < \infty$

Definition 1.16 *(Lipschutz [15]).* A subset $F \subset \mathbb{R}^n$ is *disconnected* if there exists open subsets $U, V \subset \mathbb{R}^n$ such that $F \cap U$ and $F \cap V$ are disjoint non-empty sets whose union is F.
A set is *connected* if it is not disconnected.

Definition 1.17 *(Lipschutz [15]).* A subset $F \subset \mathbb{R}^n$ is *totally disconnected* if F is the union of disjoint open subsets $U, V \subset \mathbb{R}^n$ with distinct $x, y \in F$ such that $x \in U$ and $y \in V$.

Definition 1.18 *(Falconer [8]).* The *Lebesgue measure* of $F \subset \mathbb{R}$ is

$$\mathcal{L}(F) := \inf\left\{ \sum_{i=1}^{\infty} |U_i| : \{U_i\} \text{ is a cover of } F \right\}.$$

We note that for open and closed intervals, $\mathcal{L}(a,b) = \mathcal{L}[a,b] = b-a$, so we can think of $\mathcal{L}$ as the length of a set. If $A = \bigcup_i [a_i, b_i]$, a finite/countable union of disjoint intervals, then $\mathcal{L}(A) = \sum_i (b_i - a_i)$.

2 Fractal Dimension

For the unfamiliar reader, we start with a non-rigorous definition of a fractal set. The term *fractal* was first used by Benoit Mandelbrot in 1975 to describe a group of sets that could not easily be represented through standard geometry, i.e. through functions or equations. The complexity of these objects also gave arise to the question of dimension, with many not displaying typical n-dimensional properties such as length or area. The following question is commonly used to introduce fractal qualities; "How long is the coastline of Britain?" (Mandelbrot [16]). This may seem like a simple exercise, assuming that tides do not change; take an satellite image of Britain and measure the perimeter. However, our result would be an underestimate due to the resolution excluding much of the fine detail.
Taking a greater resolution image would capture detail that we were previously blind to, and so this result would therefore be greater (arguably more accurate). But an even greater resolution image would give our length again greater than the previous two measurements. Thinking about an arbitrarily small scale, the coastline has an arbitrarily large length.
So the simple question about length cannot easily be answered as the coastline has detail on arbitrary small scales and is so irregular that traditional geometry cannot describe it.

This leads us onto our somewhat informal definition of a fractal. We say that a set F is a fractal if it has the following features:

(i) F has fine structure, i.e. detail on arbitrary small scales.

(ii) F cannot be described with traditional geometry, both locally and globally.

(iii) Often F has some sort of self-similarity, maybe approximate or statistical.

(iv) Usually, the *fractal dimension* of F is greater than its topological dimension.

(v) F is often defined in a simple way, e.g. recursively.

Due to the nature of these sets, a single definition is not universally accepted, and hence we take the term 'fractal' to be merely descriptive. An example of a fractal is the middle-thirds Cantor set which we will see in Section 2.4. We note that the example of Britain's coastline above, whilst not holding every property, is often described as a fractal.
We note the term 'fractal dimension' above. As previously mentioned, fractal sets do not always hold n-dimensional properties seen in classical geometric systems, and so we introduce a different definition of dimension. We shall focus on the *Hausdorff dimension*, but other dimensions are commonly seen and used, such as the self-similarity or box-counting dimensions. Different definitions usually give

equal values, and this is often greater than the topological dimension that the set is defined in.

Remark. Some authors refer to the Hausdorff dimension as the *Hausdorff-Besicovitch dimension.*

2.1 Measure

The Hausdorff dimension is defined through the use of measures, and many properties following it rely on results in this section. To define a measure on $\mathbb{R}^n$, we first need to introduce σ-algebra.

Definition 2.1 *(Bartle [1]).* Let X be a set and let $\mathcal{P}(X)$ be its power set. A *σ-algebra* Σ is a non-empty collection of subsets of X (i.e. $\Sigma \subseteq \mathcal{P}(X)$) such that

(σ1) $X \in \Sigma$ and $\emptyset \in \Sigma$.

(σ2) if $A \in \Sigma$, then its complement $A^c = X \backslash A \in \Sigma$.

(σ3) if (A_n) is a sequence of sets in Σ, then the union $\bigcup_i A_i \in \Sigma$.

The next proposition follows from this definition.

Proposition 2.2 *(Sengupta [22]). Let Σ be a σ-algebra of subsets of X. If (A_n) is a sequence of sets in Σ, then the intersection $\bigcap_i A_i \in \Sigma$.*

Proof. By De Morgan's law, we see that

$$\bigcap_i A_i = \left(\bigcup_i (A_i^c) \right)^c,$$

thus the result follows from (σ2) and (σ3). □

A certain type of set is needed for the definition of a measure on $\mathbb{R}^n$, namely Borel sets. These are found using the concept of a *smallest* σ-algebra. We say that a σ-algebra Σ_1 is smaller than Σ_2 if $\Sigma_1 \subset \Sigma_2$.
The smallest σ-algebra exists as a consequence of Proposition 2.2, and we can define it as the intersection of all σ-algebra containing a collection of subsets of X. We refer to this as the σ-algebra generated by a set.

Definition 2.3 *(Sengupta [22]).* Let X be a set and let $\mathcal{A}$ be a non-empty collection of subsets of X. The smallest σ-algebra containing $\mathcal{A}$ is called the *σ-algebra generated by $\mathcal{A}$.*

Definition 2.4 *(Dickson [4]).* Let Σ' be a σ-algebra generated by open sets. A *Borel set* is a set $F \in \Sigma'$.

Now we define a measure on $\mathbb{R}^n$ as follows.

Definition 2.5 *(Bartle [1]).* Let Σ' be a σ-algebra generated by open sets. A function μ from Σ' to the extended real line $\mathbb{R} \cup \{\infty\}$ is a *measure* on $\mathbb{R}^n$ if

(M1) $\mu(\emptyset) = 0$.

(M2) $\mu(F) \geq 0$ for all Borel sets F.

(M3) for all finite/countable collections of Borel sets $\{F_i\}$ with $F_i \in \Sigma'$ for all i,

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mu(F_i).$$

Equality holds if all F_i are pairwise disjoint.

Remark. A measure can take the value ∞, and so we follow the convention that $0(\pm\infty) = 0$.

We can think of a measure μ as the size of the set F measured in some way. Following are some immediate results from this definition that will be useful later.

Corollary 2.6 *(Falconer [8]). Suppose that A and B are Borel sets such that $B \subset A$. For a measure μ on $\mathbb{R}^n$,*

$$\mu(A \backslash B) = \mu(A) - \mu(B).$$

Proof. We may express A as the disjoint union $A = B \cup (A \backslash B)$. The result follows from (M3). □

Corollary 2.7 *(Falconer [8]). Suppose A_δ are Borel sets that increase as $\delta > 0$ decreases, i.e. $A_{\delta'} \subset A_\delta$ for $0 < \delta < \delta'$. Then*

$$\lim_{\delta \to 0} \mu(A_\delta) = \mu\left(\bigcup_{\delta > 0} A_\delta\right).$$

Proof. We note that

$$\bigcup_{\delta > 0} A_\delta = A_{\delta_1} \cup (A_{\delta_2} \backslash A_{\delta_1}) \cup (A_{\delta_3} \backslash A_{\delta_2}) \cup \dots$$

is pairwise disjoint for (δ_i) a decreasing sequence. By (M3) and Corollary 2.6,

$$\begin{aligned}
\mu\left(\bigcup_{\delta > 0} A_\delta\right) &= \mu(A_{\delta_1}) + \sum_{i=1}^{\infty} (\mu(A_{\delta_{i+1}}) - \mu(A_{\delta_i})) \\
&= \mu(A_{\delta_1}) + \lim_{k \to \infty} \sum_{i=1}^{k} (\mu(A_{\delta_{i+1}}) - \mu(A_{\delta_i})) \\
&= \mu(A_{\delta_1}) + \lim_{\delta_i \to 0} \sum_{\delta_i > 0} (\mu(A_{\delta_{i+1}}) - \mu(A_{\delta_i})) \\
&= \lim_{\delta \to 0} \mu(A_\delta).
\end{aligned}$$

□

For a measure μ, it is possible to think about an integral with respect to μ. For this, we introduce the definition of a simple function.

Definition 2.8 *(Edgar [5]).* A real-valued function is a *simple function* if it takes a finite number of values. For a Borel set D, a simple function $\varphi : D \to \mathbb{R}$ has the form

$$\varphi(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x)$$

with $a_i \in \mathbb{R}$ and A_i pairwise disjoint for each i, where $\mathbb{1}_{A_i}$ is the indicator function of the set A_i.

Definition 2.9 *(Bartle [1]).* Suppose that $\varphi : D \to \mathbb{R}$ is the simple function above. The *integral of φ with respect to μ* is

$$\int \varphi \, \mathrm{d}\mu := \sum_{i=1}^{n} a_i \mu(A_i).$$

Remark. This integral takes a value on the extended real line $\mathbb{R} \cup \{\infty\}$.

Integration with respect to μ holds many of the same properties as standard integration, including the ability to integrate over a specific set. The proofs of the following are omitted, however, as they distract from the current discussion.

Lemma 2.10 *(Bartle [1]). Suppose φ, ψ are simple functions as defined above, and λ a scalar. The usual properties of integration hold:*

(i)

$$\int (\varphi + \psi) \, \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\mu.$$

(ii)

$$\int \lambda\varphi \, \mathrm{d}\mu = \lambda \int \varphi \, \mathrm{d}\mu.$$

Definition 2.11 *(Bartle [1]).* Suppose that f is a non-negative μ-measurable function. The *integral of f with respect to μ* is

$$\int f \, \mathrm{d}\mu := \sup \int \varphi \, \mathrm{d}\mu,$$

where the supremum is extended over all non-negative simple functions φ satisfying $0 \leq \varphi \leq f$.

Definition 2.12 *(Bartle [1]).* Suppose that A is a Borel subset of D, then the *integral over A* is

$$\int_A f \, \mathrm{d}\mu := \int f \mathbb{1}_A \, \mathrm{d}\mu$$

for a non-negative μ-measurable function f.

Lemma 2.13 *(Bartle [1]). Suppose that f is a non-negative μ-measurable function. If $E \subseteq F$ then*

$$\int_E f \, \mathrm{d}\mu \leq \int_F f \, \mathrm{d}\mu.$$

Lemma 2.14 *(Bartle [1]). Suppose that f is a non-negative μ-measurable function. If (E_i) is a disjoint sequence whose union is E, then*

$$\int_E f \, \mathrm{d}\mu = \sum_{i=1}^{\infty} \int_{E_i} f \, \mathrm{d}\mu.$$

2.2 Hausdorff Measure

We now introduce a specific measure using the diameters of different covers. This *s-dimensional Hausdorff measure* will be used to define the Hausdorff dimension for a set. First we give the definition of a δ-cover of a set.

Definition 2.15 *(Falconer [8]).* Suppose that $\{U_i\}$ is a finite/countable collection of non-empty subsets in $\mathbb{R}^n$ that covers $F \subset \mathbb{R}^n$. The collection $\{U_i\}$ is a *δ-cover* of F if, for a given $\delta > 0$,

$$|U_i| \leq \delta$$

for all i. We let $\mathcal{U}_\delta$ be the collection of all δ-covers of F.

For any given $\delta > 0$, there are many possible δ-covers of a set (see Figure 1 for an example). Considering every possible δ-cover leads to the definition of the s-dimensional Hausdorff content of a set.

Definition 2.16 *(Falconer [8]).* Suppose $F \subset \mathbb{R}^n$ and $s \geq 0$, then the *s-dimensional Hausdorff content* is

$$\begin{aligned} \mathcal{H}^s_\delta(F) &:= \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \ \{U_i\} \text{ is a } \delta\text{-cover of } F \right\} \\ &= \inf_{\{U_i\} \in \mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\} \end{aligned}$$

where $\mathcal{U}_\delta$ is the collection of all δ-covers of F.

In words, we consider a specific δ-cover of the set F and calculate the diameter of every set U_i in this cover. We then sum these diameters to the s-th power and seek to minimise this value across all possible δ-covers.

As δ decreases, the number of sets with diameter less than δ also decreases. This leads to a fewer number of δ-covers of F, and so $\mathcal{H}^s_\delta(F)$ approaches a limit on the extended real line. This limit we shall define as follows.

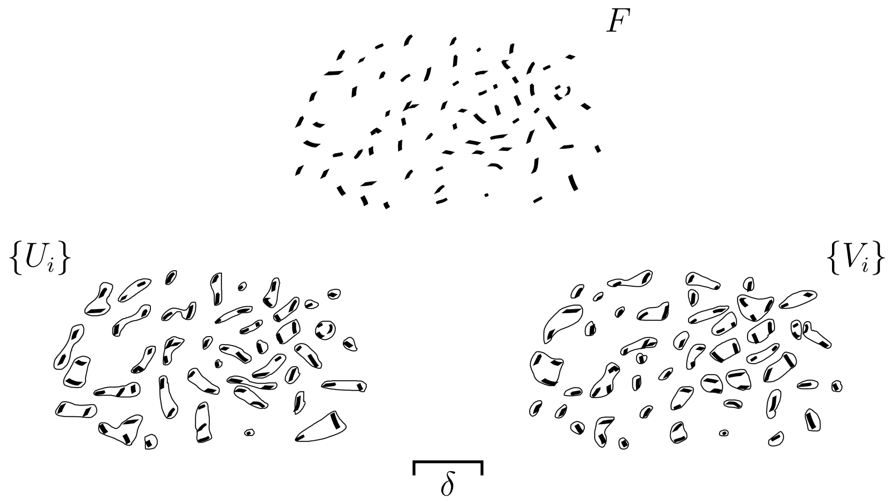

Figure 1: Two different δ-covers, $\{U_i\}$ and $\{V_i\}$, of a set F

Definition 2.17 *(Falconer [8]).* Suppose $F \subset \mathbb{R}^n$, then the *s-dimensional Hausdorff measure* is

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

Note that we have labelled $\mathcal{H}^s(F)$ a measure, and indeed the following proposition shows that this is accurate.

Proposition 2.18. *Suppose $F \subset \mathbb{R}^n$, then $\mathcal{H}^s(F)$ is a measure on $\mathbb{R}^n$.*

Proof.

(M1): For any $\delta > 0$, there exists $r > 0$ such that the ball $B_r = B(0, r^{1/s})$ is a δ-cover of $\emptyset$ with $|B_r|^s = r < \delta$. Thus

$$\mathcal{H}^s(\emptyset) = \lim_{\delta \to 0} \left(\inf_{\{B_r\} \in \mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |B_r|^s \right\} \right) = 0.$$

(M2): Let $\{U_i\}$ be an arbitrary δ-cover of F. As $|U_i| \geq 0$ for all U_i,

$$\sum_{i=1}^{\infty} |U_i|^s \geq 0$$

for all $\{U_i\} \in \mathcal{U}_\delta$, thus

$$\mathcal{H}^s(F) \geq 0.$$

(M3): Omitted, but can be shown using the Carathéodory Extension Theorem as a generalisation of the construction of the Lebesgue measure. $\square$

Further, the 1-dimensional Hausdorff measure is exactly the Lebesgue measure.

Proposition 2.19 *(Evans & Gariepy [7]). Suppose $F \subset \mathbb{R}$, then $\mathcal{H}^1(F) = \mathcal{L}(F)$.*

Proof. Suppose $F \subset \mathbb{R}^1$ and let $\delta > 0$. By the definition of the Lebesgue measure (Definition 1.18),

$$\begin{aligned}\mathcal{L}(F) &= \inf\left\{\sum_{i=1}^{\infty}|V_i| : \{V_i\} \text{ is a cover of } F\right\}\\ &\leq \inf_{\{U_i\}\in\mathcal{U}_\delta}\left\{\sum_{i=1}^{\infty}|U_i|\right\}\\ &= \mathcal{H}^1_\delta(F).\end{aligned}$$

Taking the limit,

$$\mathcal{L}(F) \leq \mathcal{H}^1(F).$$

For the reverse inequality, we let $I_k = [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$ be an interval, so

$$|V_i \cap I_k| \leq \delta$$

with

$$\sum_{k=-\infty}^{\infty}|V_i \cap I_k| \leq |V_i|.$$

We see that if $\{V_i\}$ is a cover of F, then we can take $\{V_i \cap I_k\}$ to be a δ-cover of F, thus

$$\begin{aligned}\mathcal{L}(F) &= \inf\left\{\sum_{i=1}^{\infty}|V_i| : \{V_i\} \text{ is a cover of } F\right\}\\ &\geq \inf\left\{\sum_{i=1}^{\infty}\sum_{k=-\infty}^{\infty}|V_i \cap I_k| : \{V_i\} \text{ is a cover of } F\right\}\\ &= \inf_{\{U_i\}\in\mathcal{U}_\delta}\left\{\sum_{i=1}^{\infty}|U_i|\right\}\\ &= \mathcal{H}^1_\delta(F).\end{aligned}$$

Taking the limit,

$$\mathcal{L}(F) \geq \mathcal{H}^1(F).$$

Hence $\mathcal{H}^1(F) = \mathcal{L}(F)$. □

We know that if a set is scaled by a factor λ, then its length is also scaled by this factor. The above result shows that this property holds for $\mathcal{H}^1$, so one could ask if $\mathcal{H}^s$ follows these standard scaling properties for different values of s. Indeed, we see that, similar to area being scaled by λ^2 and volume by λ^3, the s-dimensional Hausdorff measure is scaled by λ^s.

Proposition 2.20 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ and $\lambda > 0$, then*

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$$

where $\lambda F = \{\lambda x : \ x \in F\}$.

Proof. Let $\{U_i\}$ be a δ-cover of F, then $\{\lambda U_i\}$ is a $\lambda\delta$-cover of λF with

$$\begin{aligned} \mathcal{H}^s_{\lambda\delta}(\lambda F) &= \inf_{\{\lambda U_i\} \in \mathcal{U}_{\lambda\delta}} \left\{ \sum_{i=1}^{\infty} |\lambda U_i|^s \right\} \\ &\leq \sum_{i=1}^{\infty} |\lambda U_i|^s \\ &= \lambda^s \sum_{i=1}^{\infty} |U_i|^s \end{aligned}$$

as $\lambda > 0$. Taking infima,

$$\mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \lambda^s \mathcal{H}^s_{\delta}(F),$$

then taking the limit as $\delta \to 0$,

$$\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F). \tag{2.1}$$

For the reverse inequality,

$$\begin{aligned} \mathcal{H}^s(F) &= \mathcal{H}^s\left(\frac{1}{\lambda}\lambda F\right) \\ &\leq \frac{1}{\lambda^s}\mathcal{H}^s(\lambda F), \end{aligned}$$

by (2.1), so

$$\lambda^s \mathcal{H}^s(F) \leq \mathcal{H}^s(\lambda F),$$

and thus equality follows. □

2.3 Hausdorff Dimension

We can now define the Hausdorff dimension of a set. We shall see that this dimension is defined for any set in $\mathbb{R}^n$, but the actual calculation of a value is often difficult.

Considering the s-dimensional Hausdorff content, we notice that if $\delta < 1$, then $\mathcal{H}^s_\delta(F)$ is non-increasing with s for any set $F \subset \mathbb{R}^n$; this is due to $|U_i| < 1$ for all $\{U_i\} \in \mathcal{U}_\delta$. It follows that $\mathcal{H}^s(F)$ is also non-increasing with s.
We can strengthen this thought further with the following proposition.

Proposition 2.21 *(Hochman [10]). Suppose $F \subset \mathbb{R}^n$ and let $t > s$.*

(i) If $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$.

(ii) If $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$.

Proof. Consider a δ-cover $\{U_i\} \in \mathcal{U}_\delta$ and let $t > s$, then

$$\begin{aligned}\mathcal{H}^t_\delta(F) &= \inf_{\{U_i\}\in\mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |U_i|^t \right\} \\ &\leq \sum_{i=1}^{\infty} |U_i|^t \\ &= \sum_{i=1}^{\infty} |U_i|^{t-s}|U_i|^s \\ &\leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s\end{aligned}$$

as $|U_i| \leq \delta$. Taking infima,

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$

Letting $\delta \to 0$,

$$\mathcal{H}^s(F) < \infty \implies \mathcal{H}^t(F) = 0$$

and

$$\mathcal{H}^t(F) > 0 \implies \mathcal{H}^s(F) = \infty.$$ □

Looking at the plot of $\mathcal{H}^s$ against s (Figure 2), we see that there exists a unique value of s for which $\mathcal{H}^t(F) = \infty$ when $t < s$ and $\mathcal{H}^t(F) = 0$ when $t > s$. This 'jump' point s is defined as the Hausdorff dimension of the corresponding set F.

Definition 2.22 *(Falconer [8]).* For a set $F \subset \mathbb{R}^n$, its *Hausdorff dimension* is

$$\dim_H F := \sup\{s : \ \mathcal{H}^s(F) = \infty\} = \inf\{s \geq 0 : \ \mathcal{H}^s(F) = 0\}.$$

At $s = \dim_H F$, we note that $0 \leq \mathcal{H}^s(F) \leq \infty$. If F is a Borel set and $0 < \mathcal{H}^s(F) < \infty$, we say that F is an *s-set*.
Calculating this jump point is often difficult to do, so in many examples we try to

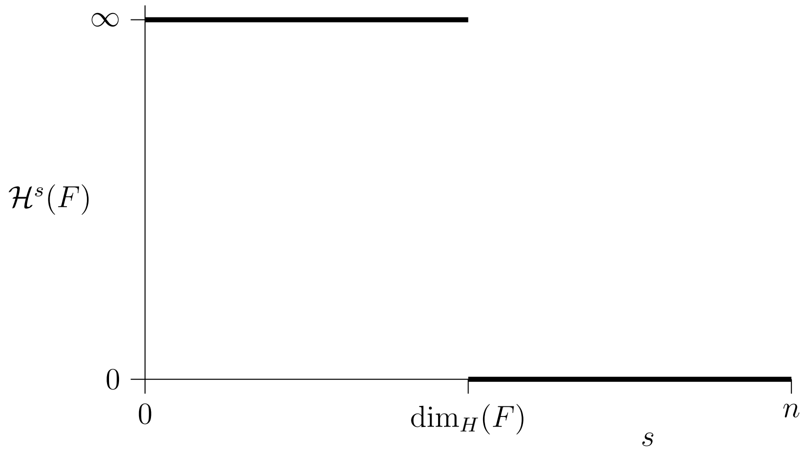

Figure 2: $\mathcal{H}^s(F)$ against s for a set F

find a positive non-zero upper and lower bound of $\mathcal{H}^s$ at a specific s. This shows us that $\mathcal{H}^s$ is not infinite or 0, and so s must be the Hausdorff dimension of the set. As noted above, an upper and lower bound cannot always be found, but many fractal sets we consider do have these bounds.

Before calculating examples, we shall explore some properties of the Hausdorff dimension. Similar to the scaling properties above, we can consider sets that have been transformed in some way, using functions for example. First we look at those mappings that satisfy a Hölder condition for exponent α (Definition 1.14).

Proposition 2.23 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$ be a mapping that satisfies a Hölder condition for exponent α. Then*

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F)$$

for every $s \geq 0$ with constant $c > 0$.

Proof. Let $\{U_i\} \in \mathcal{U}_\delta$. If f satisfies a Hölder condition for exponent α, then

$$|f(F \cap U_i)| \leq c|U_i|^\alpha$$

for constant $c > 0$ as $\{U_i\}$ is a cover of F. It follows that $\{f(F \cap U_i)\}$ is a δ'-cover of $f(F)$ with $\delta' = c\delta^\alpha$, and thus

$$\sum_i |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum_i |U_i|^s,$$

so

$$\mathcal{H}^{s/\alpha}_{\delta'}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s_\delta(F).$$

Letting $\delta \to 0$ implies $\delta' \to 0$, hence

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F).$$ □

Proposition 2.24 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$ be a mapping that satisfies a Hölder condition for exponent α, then*

$$\dim_H f(F) \leq \frac{1}{\alpha} \dim_H F.$$

Proof. We consider $s > \dim_H F$, so $\mathcal{H}^s(F) = 0$ by the definition of the Hausdorff dimension. By Proposition 2.23 above,

$$0 \leq \mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F) = 0,$$

so $\mathcal{H}^{s/\alpha}(f(F)) = 0$. Thus

$$\dim_H f(F) \leq \frac{s}{\alpha} \leq \frac{1}{\alpha} \dim_H F$$

for all $s \geq 0$. □

Letting $\alpha = 1$, we can consider mappings that satisfy the Lipschitz condition.

Corollary 2.25 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$ be a Lipschitz mapping, then*

$$\dim_H f(F) \leq \dim_H F.$$

Proof. Immediate from Proposition 2.24 with $\alpha = 1$. □

Corollary 2.26 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$ be a bi-Lipschitz mapping, then*

$$\dim_H f(F) = \dim_H F.$$

Proof. Immediate from Proposition 2.24 and Corollary 2.25 using f^{-1} and $\alpha = 1$. □

We conclude this section by using the results above to explore a link between the Hausdorff dimension and the connectivity of a set.

Theorem 2.27 *(Falconer [8, Proposition 2.5]). A subset $F \subset \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected.*

Before proceeding with the proof of this theorem, we need to first introduce the following lemma.

Lemma 2.28. *Suppose $F \subset \mathbb{R}^1$. If*

$$\mathcal{H}^1(F) = \lim_{\delta \to 0} \left(\inf_{\{U_i\} \in \mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |U_i| \right\} \right) = 0,$$

then F has a dense complement in $\mathbb{R}$.

Proof. We want to show that $F^c \subset \mathbb{R}$ is dense, i.e. for each $x \in \mathbb{R}$ and all $\varepsilon > 0$, there exists $z \in F^c$ such that $|z-x| < \varepsilon$ (Definition 1.11). Clearly, this is equivalent to showing that for all $\varepsilon > 0$, there exists $z \in (x-\varepsilon, x+\varepsilon) =: I_\varepsilon$.
We know

$$\mathcal{H}^1(I_\varepsilon) = \mathcal{L}(I_\varepsilon) = 2\varepsilon$$

by Proposition 2.19 and by the definition of the Lebesgue measure, so the restriction

$$\begin{aligned}\mathcal{H}^1(F^c \cap I_\varepsilon) &= \mathcal{H}^1(I_\varepsilon \backslash F) \\ &= \mathcal{H}^1(I_\varepsilon) - \mathcal{H}^1(F) \\ &= \mathcal{H}^1(I_\varepsilon) = 2\varepsilon\end{aligned}$$

by Corollary 2.6 and as $\mathcal{H}^1(F) = 0$. Since $\varepsilon > 0$, it follows that $F^c \cap I_\varepsilon$ has a non-zero positive $\mathcal{H}^1$ measure, and thus is non-empty. Hence for each $x \in \mathbb{R}$ and all $\varepsilon > 0$, there exists $z \in F^c \cap (x-\varepsilon, x+\varepsilon)$, so F^c is dense is $\mathbb{R}$. □

Proof of Theorem 2.27. Let $x, y \in F$ be distinct points. We aim to show that x and y lie in two different disjoint open sets, the union of which is F.

Define the mapping $f : \mathbb{R}^n \to [0, \infty)$ by $f(z) = |z-x|$, then

$$\begin{aligned}|f(z) - f(w)| &= ||z-x| - |w-x|| \\ &\leq |(z-x) - (w-x)| \\ &= |z-w|\end{aligned}$$

by the reverse triangle inequality, so f is Lipschitz. By Corollary 2.25,

$$\dim_H f(F) \leq \dim_H F < 1$$

where $f(F) \subset \mathbb{R}$ as f projects onto $[0, \infty)$. Thus $\mathcal{H}^1(f(F)) = 0$ as $\mathcal{H}^s(f(F)) = 0$ for all $s > \dim_H f(F)$. By Lemma 2.28, $f(F)$ has a dense complement in $\mathbb{R}$.
Choosing $r \notin f(F)$ (i.e. in the complement of $f(F)$) such that $0 < r < f(y)$, it follows that

$$F = \{z \in F : \ |z-x| < r\} \cup \{z \in F : \ |z-x| > r\}.$$

Thus, as r is contained in a dense set, F is the union of two disjoint open sets such that $x \in \{z \in F : \ |z-x| < r\}$ and $y \in \{z \in F : \ |z-x| > r\}$. Hence, as x and y were arbitrary and distinct, F is totally disconnected. □

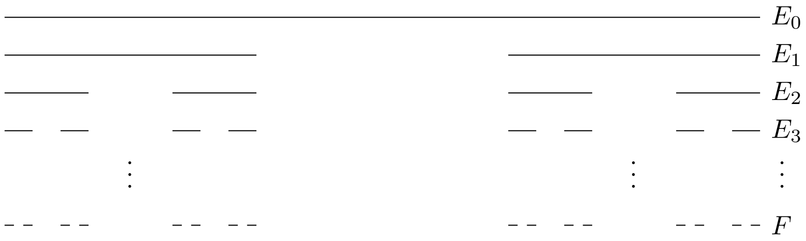

Figure 3: The middle-thirds Cantor set F

2.4 The Middle-Thirds Cantor Set

We now look to calculate the Hausdorff dimension of a specific set, namely the *middle-thirds Cantor set*. This fractal set is named after the German mathematician Georg Cantor, who abstractly described the set in 1883. It is usually constructed through an iterative process on $\mathbb{R}$, as below.

We begin with the closed unit interval $E_0 = [0, 1]$. Removing the open middle-thirds of the interval, we get

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Repeating this removal for the remaining two intervals, we get

$$E_2 = \left\{ \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \right\} \cup \left\{ \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \right\}.$$

Continuing this process of removing the open middle-thirds of the remaining intervals, we see that at the k-th stage of construction, E_k is the union of 2^k intervals each of length $\frac{1}{3^k}$; we label these as the level-k intervals. The limit as this process is repeated an infinite number of times is the middle-thirds Cantor set F (see Figure 3). Formally, we define F as the intersection of every E_k,

$$F = \bigcap_{k=0}^{\infty} E_k.$$

Comparing F to the fractal set conditions, one can see that most are satisfied. Before continuing with the thought of dimension, we first introduce two propositions that will be helpful in its calculation.

Proposition 2.29. *The middle-thirds Cantor set F is closed.*

Proof. As $F = \bigcap_{k=0}^{\infty} E_k$, the result follows from Proposition 1.9. $\square$

Proposition 2.30. *The middle-thirds Cantor set F is compact.*

Proof. Clearly $F = \bigcap_{k=0}^{\infty} E_k$ is bounded as $E_0 = [0,1]$. The result follows from the Heine-Borel theorem. □

As mentioned above, calculating the Hausdorff dimension is difficult as we need to consider every possible δ-cover of a set, so we use the knowledge that, for many sets, $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_H F$. Thus we aim to find a non-zero positive upper and lower bound of $\mathcal{H}^s(F)$ for a specific s, deducing that $\dim_H F$ is equal to this s.

Upper bound

Finding an upper bound to $\mathcal{H}^s(F)$ is usually quite straight forward as we need only consider a single cover of the set F. We notice that for each E_k, there are 2^k level-k intervals of length $\frac{1}{3^k}$. Considering the 'natural' cover using these intervals, we find a $\delta = 3^{-k}$-cover of F. It follows that

$$\mathcal{H}^s_\delta(F) = \mathcal{H}^s_{3^{-k}}(F) = \inf_{\{U_i\}\in\mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\} \leq \sum_{i=1}^{2^k} \left(\frac{1}{3^k}\right)^s = 2^k(3^{-k})^s.$$

We want to consider when $\delta \to 0$, so letting $k \to \infty$, we find

$$\mathcal{H}^s(F) \leq \lim_{k\to\infty} 2^k(3^{-s})^k.$$

We wish for a non-zero upper bound, so letting $s = \frac{\log 2}{\log 3}$,

$$\begin{aligned} \mathcal{H}^s(F) &\leq \lim_{k\to\infty} 2^k \left(3^{\frac{\log 2}{\log 3}}\right)^k \\ &= \lim_{k\to\infty} \left(2 \cdot \frac{1}{2}\right)^k \\ &= 1. \end{aligned}$$

Hence when $s = \frac{\log 2}{\log 3}$, we have found an upper bound of $\mathcal{H}^s(F)$.

Lower bound

Finding a lower bound is a lot trickier as one must find a bound for all δ-covers of F. To help our case, we reduce the number of sets that we need to consider. We do this by showing that for two covers $\{U_i\}$ and $\{V_i\}$ where $\{U_i\} \subset \{V_i\}$, if

$$M \leq \sum_i |U_i|^s \tag{2.2}$$

for some $M > 0$, then

$$M \leq \sum_i |V_i|^s.$$

First we narrow down the covers of F to a collection of intervals.

Let $\{V_i\}$ be a collection of sets that cover F, and let $\{W_i\}$ be the collection of intervals such that

$$W_i = [\inf V_i, \sup V_i]$$

for each i. If $\{W_i\}$ satisfies (2.2), then

$$M \leq \sum_i |W_i|^s = \sum_i |V_i|^s$$

as $|W_i| = |V_i|$. Thus it is enough to assume any cover of F is by a collection of intervals.

We can narrow this down further, showing that it is enough to assume any cover of F is by a finite collection of closed intervals.

Consider $\{W_i\}$, a collection of intervals that cover F such that W_i is the interval from a_i to b_i. Expanding each interval slightly such that

$$W_i' = \left(a_i - \frac{\varepsilon}{2^i}, b_i + \frac{\varepsilon}{2^i}\right)$$

for $\varepsilon > 0$, we get an open cover of F, namely $\{W_i'\}$. By Proposition 2.30, F is compact, and so by the definition of compactness (Definition 1.5) there exists a finite subcover $\{W_1', W_2', \ldots, W_N'\}$. We take the closure

$$U_j = \overline{W_j'}$$

for $1 \leq j \leq N$ such that $\{U_j\}$ is a finite collection of closed intervals. Thus if $\{U_j\}$ satisfies (2.2), then

$$\begin{aligned}
M &\leq \sum_{j=1}^{N} |U_j|^s \\
&= \sum_{j=1}^{N} |\overline{W_j'}|^s \\
&\leq \sum_{i=1}^{\infty} |W_i'|^s \\
&= \sum_{i=1}^{\infty} \left(|W_i| + \frac{2\varepsilon}{2^i}\right)^s
\end{aligned}$$

by Proposition 1.13. Since $\varepsilon > 0$ can be chosen arbitrarily,

$$\sum_{i=1}^{\infty} \left(|W_i| + \frac{2\varepsilon}{2^i}\right)^s \to \sum_{i=1}^{\infty} |W_i|^s$$

as $\varepsilon \to 0$ (see Proposition A.1). Hence

$$M \leq \sum_{i=1}^{\infty} |W_i|^s$$

for any collection of intervals $\{W_i\}$ that cover F. So it is sufficient to assume any cover of F is by a finite collection of closed intervals.

For any of these finite δ-covers $\{U_i\} \in \mathcal{U}_{\delta_{\text{fin}}}$, we categorise each U_i as follows; let k be the integer such that

$$3^{-(k+1)} \leq |U_i| < 3^{-k}, \tag{2.3}$$

so U_i intersects only one level-k interval of E_k (as the separation distance of the level-k intervals is at least 3^{-k}).
Consider now the intervals of E_j for $j \geq k$; we see that U_i can only intersect at most 2^{j-k} level-j intervals as each interval 'splits' into two at the next stage of construction. Letting $s = \frac{\log 2}{\log 3}$,

$$\begin{aligned}
2^{j-k} &= 2^j 2^{-k} \\
&= 2^j 3^{-sk} \\
&= 2^j (3^{-k})^s \\
&\leq 2^j (3|U_i|)^s && \text{by (2.3)} \\
&= 2^j 3^s |U_i|^s.
\end{aligned}$$

This shows us that the number of level-j intervals that U_i intersects is at most $2^j 3^s |U_i|^s$. The whole δ-cover $\{U_i\}$ intersects every interval of E_j, and since there are 2^j intervals of length 3^{-j},

$$\begin{aligned}
&& 2^j &\leq \sum_{i=1}^{N} 2^j 3^s |U_i|^s \\
\implies && 3^{-s} &\leq \sum_{i=1}^{N} |U_i|^s \\
\implies && \frac{1}{2} &\leq \sum_{i=1}^{N} |U_i|^s
\end{aligned}$$

for all $\{U_i\} \in \mathcal{U}_{\delta_{\text{fin}}}$. Thus $\frac{1}{2} \leq \mathcal{H}^s_\delta(F)$, and so taking limits,

$$\frac{1}{2} \leq \mathcal{H}^s(F),$$

a non-zero positive lower bound when $s = \frac{\log 2}{\log 3}$.

We conclude

$$\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$$

when $s = \frac{\log 2}{\log 3}$, and thus by the definition of the Hausdorff dimension,

$$\dim_H F = s = \frac{\log 2}{\log 3}.$$

The middle-thirds Cantor set F therefore has a Hausdorff dimension of approximately 0.631, so using the following corollary, we see that F does not possess properties of a typical one-dimensional object, such as length.

Corollary 2.31. *The middle-thirds Cantor set $F \subset \mathbb{R}$ is totally disconnected.*

Proof. As $\dim_H F = \frac{\log 2}{\log 3} < 1$, we see that F is totally disconnected by Proposition 2.27. □

3 The Mass Distribution Principle

It can be quite tedious to calculate a lower bound of $\mathcal{H}^s$, as we saw with the middle-thirds Cantor set. For a lot of sets, we are not able to reduce the cover down to a nice workable collection. We introduce another way to find the lower bound using measures, specifically mass distributions defined below.

Definition 3.1 *(Falconer [8]).* The *support* of a measure μ is the smallest closed set X such that $\mu(\mathbb{R}^n \backslash X) = 0$.
We say that μ is a measure *on* a set A if A contains the support of μ.

Definition 3.2 *(Falconer [8]).* A *mass distribution* μ is a measure on a bounded subset of $\mathbb{R}^n$ such that $0 < \mu(\mathbb{R}^n) < \infty$.

As the name suggests, we usually think of this measure as a finite mass spread across a set in some way, giving us a mass distribution that satisfies the conditions of a measure (Definition 2.5).
Thinking back to finding a lower bound of $\mathcal{H}^s$, we need to consider all possible covers of a set. Using a mass distribution μ, we show that if each set U in a δ-cover has a measure less than its size $|U|^s$ to a scale factor, then a lower bound of $\mathcal{H}^s$ can be found.
The mass distribution principle formalises this idea, and compares the mass $\mu(U)$ with $|U|^s$ for each U.

Theorem 3.3 *(**Mass distribution principle**, Falconer [8, Theorem 4.2]). Let μ be a mass distribution on $F \subset \mathbb{R}^n$. Suppose that for some s, there exists $c > 0$ and $\delta > 0$ such that*

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \delta$. Then

$$\frac{\mu(F)}{c} \leq \mathcal{H}^s(F)$$

and $s \leq \dim_H F$.

Proof. Suppose that $\{U_i\}$ is any cover of F satisfying the conditions above, then

$$0 < \mu(F) = \mu\left(\bigcup_i U_i\right) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s$$

by (M3). Taking infima,

$$\mu(F) \leq c\mathcal{H}^s_\delta(F)$$

for sufficiently small δ, and so taking limits,

$$\frac{\mu(F)}{c} \leq \mathcal{H}^s(F).$$

It follows that $s \leq \dim_H F$ by definition. □

This result provides us with a useful connection between measure and the dimension of a set, and can be use in many examples to find the Hausdorff dimension.

3.1 The Von Koch Curve

Using the mass distribution principle above, we aim to find the Hausdorff dimension of an example fractal, namely the *von Koch curve*. First constructed in 1904 by Helge von Koch, the curve is one of the earliest fractal shapes to be described and is often seen in a snowflake arrangement. It is constructed through an iterative process similar to that of the middle-thirds Cantor set.

Starting with $E_0 = [0, 1]$ in $\mathbb{R}^2$, we replace the middle-thirds of the line with two sides of an equilateral triangle. We label this E_1 (see Figure 4) and again replace the middle-thirds of the four lines with two sides of an equilateral triangle. The von Koch curve F is the limit approached by E_k as $k \to \infty$. We note that at the k-th stage of construction, E_k is the union of 4^k lines each of length $\frac{1}{3^k}$.
To calculate the Hausdorff dimension of F, we first use a natural cover to find an upper bound of $\mathcal{H}^s(F)$ for a specific s. Then, using the mass distribution principle, we obtain a non-zero positive lower bound.

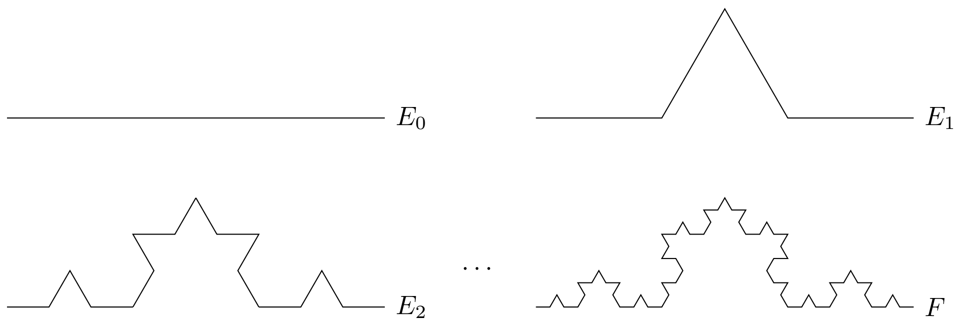

Figure 4: The von Koch curve F

Upper bound

Consider the natural cover of E_k consisting of 4^k lines each of length $\frac{1}{3^k}$. Letting $\delta = 3^{-k}$,

$$\mathcal{H}^s_\delta(F) = \inf_{\{U_i\}\in\mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\} \leq \sum_{i=1}^{4^k} \left(\frac{1}{3^k}\right)^s = 4^k(3^{-sk}).$$

We see that $\delta \to 0$ as $k \to \infty$, so

$$\mathcal{H}^s(F) \leq \lim_{k\to\infty} 4^k(3^{-sk}).$$

Letting $s = \frac{\log 4}{\log 3}$,

$$\begin{aligned} \mathcal{H}^s(F) &\leq \lim_{k\to\infty} \left(4 \cdot 3^{-\frac{\log 4}{\log 3}}\right)^k \\ &= \lim_{k\to\infty} \left(4 \cdot \frac{1}{4}\right)^k \\ &= 1, \end{aligned}$$

an upper bound.

Lower bound

Now we consider a mass distribution μ on F such that $\mu(F) = 1$. Starting with the unit mass on E_0, we split the mass such that each of the 4^k lines of E_k has an equal mass of $\frac{1}{4^k}$. Similar to the lower bound of the middle-thirds Cantor set, we categorise each U_i in a δ-cover of F; let k be the integer such that

$$3^{-(k+1)} \leq |U_i| < 3^{-k}, \tag{3.1}$$

so U_i intersects at most two lines of E_k. As each line of E_k has a mass of $\frac{1}{4^k}$, by (M3),

$$\begin{aligned}\mu(U_i) &\le 2 \cdot \frac{1}{4^k} \\ &= 2\left(3^{-\frac{\log 4}{\log 3}}\right)^k \\ &= 2\left(3^{-k}\right)^{\frac{\log 4}{\log 3}} \\ &< 2|U_i|^{\frac{\log 4}{\log 3}}\end{aligned}$$

by (3.1). Hence, using the mass distribution principle,

$$\mathcal{H}^{\frac{\log 4}{\log 3}}(F) \ge \frac{\mu(F)}{2} = \frac{1}{2},$$

a non-zero positive lower bound.

Thus, by the definition of the Hausdorff dimension,

$$\dim_H F = \frac{\log 4}{\log 3}.$$

We see that, although not entirely straight forward, using the mass distribution principle makes calculating a lower bound significantly easier than the previous method.

3.2 General Cantor Sets

The mass distribution principle allows us to consider the dimension of a more general class of sets, *general Cantor sets* (we simply refer to these as *Cantor sets*). Constructed in a similar manner to the middle-thirds Cantor set, we start with the unit interval $E_0 = [0, 1]$. Instead of each level-k interval being split into 2 intervals of length scaled by $\frac{1}{3}$, we consider them to be split into $m \ge 2$ intervals of length scaled by λ. If between each stage of construction m is not constant, then each subsequent interval has differing lengths, and we discuss Cantor sets with this property first.

Consider a Cantor set F such that at each stage of its construction, the level-k intervals are equally spaced and have equal lengths. Let s be a number strictly between 0 and 1. Assume F is constructed such that at the k-th stage of construction, E_k has the following property: for each level-k interval I of E_k, the level-$k+1$ subintervals

$$I_1, I_2, \ldots, I_m$$

of I, for $m \in \mathbb{Z}$ with $m \geq 2$, are equally spaced and have equal length satisfying

$$|I_i|^s = \frac{1}{m}|I|^s. \tag{3.2}$$

We let the left-hand ends of I_1 and I coincide, along with the right-hand ends of I_m and I.

Example 3.1. Consider the middle-thirds Cantor set with $m = 2$ and let $s = \frac{\log 2}{\log 3} \in (0, 1)$, then

$$|I_i|^s = \left(\frac{1}{3}|I|\right)^{\frac{\log 2}{\log 3}} = \frac{1}{2}|I|^s,$$

satisfying (3.2).

Using this general construction, we can now calculate the Hausdorff dimension for these Cantor sets.

Proposition 3.4 *(Falconer [8]). Suppose that F is a Cantor set constructed with E_k as above, then F is an s-set with $0 < \mathcal{H}^s(F) < \infty$ and*

$$\dim_H F = s.$$

Proof. We first find an upper bound of $\mathcal{H}^s(F)$ using a natural covering, then we distribute a mass on F and use the mass distribution principle to obtain a non-zero positive lower bound.

Upper bound

For each interval I of E_k,

$$|I|^s = \sum_{i=1}^{m} |I_i|^s,$$

and, inductively,

$$\sum |I_i|^s = 1$$

over all intervals I. The intervals of E_k cover F, and since the maximum interval length tends to 0 as $k \to \infty$,

$$\mathcal{H}^s_\delta(F) \leq 1$$

for δ sufficiently small. Thus taking limits,

$$\mathcal{H}^s(F) \leq 1 < \infty,$$

an upper bound.

Lower bound

We distribute a unit mass μ on F such that for a level-k interval I,

$$\mu(I) = |I|^s.$$

We can think of this distribution as dividing a unit mass on $[0,1]$ equally between each interval of E_1; their mass subsequently being equally divided between the subintervals on E_2, and so on. The property $|I|^s = \sum_{i=1}^{m} |I_i|^s$ ensures that $\mu(I) = |I|^s$. For any interval U with end points in F, we aim to estimate $\mu(U)$.
Let I be the smallest level-k interval containing U, with $I_1, \ldots, I_m$ subintervals of I in E_{k+1}. The spacing between consecutive I_i is

$$\begin{aligned}
\frac{1}{m-1}(|I| - m|I_i|) &= |I|\frac{1 - m\frac{|I_i|}{|I|}}{m-1} \\
&= |I|\frac{1 - m(m^{-1/s})}{m-1} && \text{by (3.2)} \\
&\geq |I|\frac{1 - 2^{1-1/s}}{m-1} && \text{as } m \geq 2 \\
&\geq |I|\frac{1 - 2^{1-1/s}}{m} \\
&= c_s\frac{|I|}{m}
\end{aligned}$$

for $c_s = 1 - 2^{1-1/s}$.
The interval U intersects j of the I_i intervals, with $j \geq 2$, otherwise U would be contained within a smaller interval of E_{k+1}. So U crosses $j-1$ gaps between the j intervals, thus

$$|U| \geq \frac{j-1}{m}c_s|I| \geq \frac{j}{2m}c_s|I| \tag{3.3}$$

as $j - 1 \geq \frac{j}{2}$. Hence

$$\begin{aligned}
\mu(U) &\leq j\mu(I_i) \\
&= j|I_i|^s \\
&= \frac{j}{m}|I|^s \\
&\leq 2^s c_s^{-s}\left(\frac{j}{m}\right)^{1-s}|U|^s && \text{by (3.3)} \\
&\leq 2^s c_s^{-s}|U|^s
\end{aligned}$$

as $j \leq m$ and $0 \leq s \leq 1$, so $\left(\frac{j}{m}\right)^{1-s} \leq 1$.
This holds for any interval U with endpoints in F, and so it holds for any set U (see Section 2.4). By the mass distribution principle,

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{2^s c_s^{-s}} = \frac{1}{2^s(1 - 2^{1-1/s})} > 0,$$

hence

$$0 < \mathcal{H}^s(F) < \infty$$

and

$$\dim_H F = s. \qquad \square$$

As noted above, the value of m can change throughout the construction of these Cantor sets, but what if we consider m to be fixed throughout the construction? We define Cantor sets with a fixed m as *uniform Cantor sets*, and using the above proposition we can find an explicit formula for the dimension of such a set.

Proposition 3.5 *(Falconer [8]). Let $m \geq 2$ be an integer and $0 < \lambda < \frac{1}{m}$. Let F be the set obtained by the construction in which each interval I is replaced by m equally spaced subintervals of lengths $\lambda|I|$, the ends of I coinciding with the ends of the extreme subintervals. Then*

$$\dim_H F = \frac{\log m}{-\log \lambda}$$

and $0 < \mathcal{H}^{\log m/-\log \lambda}(F) < \infty$.

Proof. Using the construction at the start of this section, we let m be constant and

$$s = \frac{\log m}{-\log \lambda}.$$

Then (3.2) is satisfied by $(\lambda|I|)^s = \frac{1}{m}|I|^s$, so $\dim_H F = s$ by Proposition 3.4. $\square$

Remark. Readers familiar with fractal sets may notice that the *similarity dimension*

$$\dim_S F := \frac{\log n}{-\log r}$$

is equal to the Hausdorff dimension for uniform Cantor sets, where at each stage of construction there are n copies of the original set scaled by a factor of r .

To conclude this section, we verify that the dimension found in Section 2.4 for the middle-thirds Cantor set coincides with the formula above. We also consider another non-trivial example.

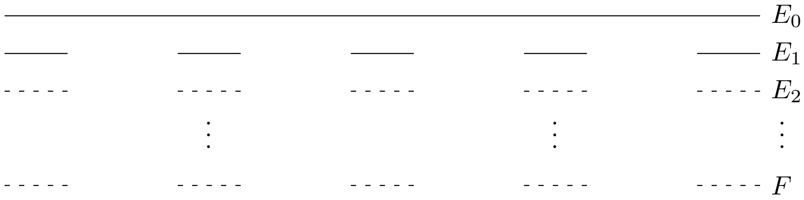

Figure 5: A uniform Cantor set F with $m = 5$, $\lambda = \frac{1}{12}$

Example 3.2. Consider the middle-thirds Cantor set F described as a uniform Cantor set with $m = 2$ and $\lambda = \frac{1}{3}$, then

$$\dim_H F = \frac{\log 2}{-\log\left(\frac{1}{3}\right)} = \frac{\log 2}{\log 3}.$$

Example 3.3. Consider the uniform Cantor set F with $m = 5$ and $\lambda = \frac{1}{12}$ (see Figure 5), then

$$\dim_H F = \frac{\log 5}{-\log\left(\frac{1}{12}\right)} = \frac{\log 5}{\log 12}.$$

3.3 Borel Sets

We now shift our focus to measures and dimensions of Borel sets (Definition 2.4). The results in this section set up later discussions on the *potential* of a set, which ultimately leads onto projections.
To begin, we consider some properties of the Hausdorff dimension of Borel sets.

Proposition 3.6 *(Edgar [6]). Suppose that A and B are Borel sets.*

(i) *If $A \subseteq B$, then $\dim_H A \leq \dim_H B$.*

(ii) $\dim_H(A \cup B) = \max\{\dim_H A, \dim_H B\}$.

Proof.

(i): Suppose $A \subseteq B$ and let $s > \dim_H B$. We see that, as a δ-cover of B is also a δ-cover of A,

$$\mathcal{H}^s(A) \leq \mathcal{H}^s(B) = 0$$

for all $s > \dim_H B$, thus $\dim_H A \leq \dim_H B$.

(ii): Since $A, B \subseteq A \cup B$,

$$\dim_H(A \cup B) \geq \max\{\dim_H A, \dim_H B\}$$

by (i). Let
$$s > \max\{\dim_H A, \dim_H B\}$$
such that $s > \dim_H A$ and $s > \dim_H B$, and thus $\mathcal{H}^s(A) = 0$ and $\mathcal{H}^s(B) = 0$. By (M3),
$$\mathcal{H}^s(A \cup B) \leq \mathcal{H}^s(A) + \mathcal{H}^s(B) = 0$$
for all $s > \max\{\dim_H A, \dim_H B\}$, therefore
$$\dim_H(A \cup B) \leq \max\{\dim_H A, \dim_H B\}.$$
Hence $\dim_H(A \cup B) = \max\{\dim_H A, \dim_H B\}$. □

Before moving onto an important proposition concerning mass distributions on Borel sets, we first need to introduce the Covering Lemma.

Lemma 3.7 ***(Covering Lemma**, Falconer [8]). Let $\mathcal{C}$ be a family of balls contained in some bounded region of $\mathbb{R}^n$. There exists a finite/countable disjoint subcollection $\{B_i\}$ such that*
$$\bigcup_{B \in \mathcal{C}} B \subset \bigcup_i \tilde{B}_i \tag{3.4}$$
where $\tilde{B}_i$ is the closed ball concentric with B_i and of four times the radius.

Proof. We prove only for a finite family $\mathcal{C}$, but the general case follows a similar argument.
We select $\{B_i\}$ inductively. Let B_1 be the ball of maximum radius in $\mathcal{C}$. Now assume $B_1, \dots, B_k$ have been chosen. We take B_{k+1} to be the ball of maximal radius in $\mathcal{C}$ such that it is disjoint from all $B_1, \dots, B_k$ already chosen. We terminate the process when no such ball remains.
Every ball chosen is disjoint by construction, so it remains to show that (3.4) holds. For $B \in \mathcal{C}$, we either have that $B = B_i$ for some i, or B intersects at least one B_i with $|B_i| \geq |B|$ (otherwise B would have been chosen instead). Thus we have that $B \subset \tilde{B}_i$. □

Remark. We notice that for a finite family $\mathcal{C}$, this lemma can be strengthened using balls of only three times the radius of B.

Using this, we can consider the mass distribution of balls centred on a Borel set as their radii decrease, linking their behaviour to the Hausdorff measure.

Proposition 3.8 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ is a Borel set. Let μ be a mass distribution on $\mathbb{R}^n$, and let $0 < c < \infty$ be a constant.*

(i) If

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} < c$$

for all $x \in F$, then

$$\frac{\mu(F)}{c} \leq \mathcal{H}^s(F).$$

(ii) If

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} > c$$

for all $x \in F$, then

$$\mathcal{H}^s(F) \leq 2^s \frac{\mu(\mathbb{R}^n)}{c}.$$

Proof.

(i): For each $\delta > 0$, let

$$F_\delta := \{x \in F : \ \mu(B(x,r)) < (c-\varepsilon)r^s \text{ for all } 0 < r \leq \delta\}.$$

Let $\{U_i\}$ be a δ-cover of F, and thus a δ-cover of F_δ also. We see that if $x \in F_\delta$ is contained in U_i, then the ball $B = B(x, |U_i|)$ contains the whole of U_i. By the definition of F_δ,

$$\mu(U_i) \leq \mu(B) < c|U_i|^s,$$

so

$$\mu(F_\delta) \leq \sum_i \{\mu(U_i) : \ U_i \text{ intersects } F_\delta\} \leq c \sum_i |U_i|^s.$$

Since $\{U_i\}$ is an arbitrary δ-cover of F, it follows that

$$\mu(F_\delta) \leq c\mathcal{H}^s_\delta(F) \leq c\mathcal{H}^s(F).$$

As $\delta \to 0$, F_δ increases to F, so

$$\mu(F) \leq c\mathcal{H}^s(F)$$

by Corollary 2.7.

(ii): We prove a weaker version of (ii), replacing 2^s with 8^s, but the basic idea of the proof is similar.
First suppose that F is bounded. Fix $\delta > 0$ and let

$$\mathcal{C} := \{B(x,r) : \ x \in F, \ 0 < r \leq \delta, \text{ and } \mu(B(x,r)) > cr^s\}$$

be a collection of balls. As $\mu(B(x,r))/r^s > c$ for all $x \in F$, then $F \subset \bigcup_{B\in\mathcal{C}} B$. Applying the Covering Lemma to $\mathcal{C}$, there exists a sequence of disjoint balls $B_i \in \mathcal{C}$ such that

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i$$

where $\tilde{B}_i$ is the closed ball concentric with B_i of four times the radius. Thus $\{\tilde{B}_i\}$ is an 8δ-cover of F, so

$$\begin{aligned}\mathcal{H}^s_{8\delta}(F) &\leq \sum_i |\tilde{B}_i|^s \\ &\leq 4^s \sum_i |B_i|^s \\ &\leq 8^s c^{-1} \sum_i \mu(B_i) \\ &\leq 8^s c^{-1} \mu(\mathbb{R}^n).\end{aligned}$$

Taking the limit as $\delta \to 0$,

$$\mathcal{H}^s(F) \leq 8^s \frac{\mu(\mathbb{R}^n)}{c} < \infty. \tag{3.5}$$

Now suppose that F is unbounded and assume for contradiction that

$$\mathcal{H}^s(F) > 8^s c^{-1} \mu(\mathbb{R}^n).$$

The $\mathcal{H}^s$-measure of a bounded subset of F is greater than this value, contradicting (3.5), hence

$$\mathcal{H}^s(F) \leq 8^s \frac{\mu(\mathbb{R}^n)}{c}$$

for unbounded F. □

Further exploring properties of Borel sets yields an interesting result concerning the existence of a compact subset when the Hausdorff measure is infinite.

Theorem 3.9 *(Falconer [8, Theorem 4.10]). Suppose that F is a Borel subset of $\mathbb{R}^n$ with $\mathcal{H}^s(F) = \infty$. There exists a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$.*

Proof. Omitted as we need only concern ourselves with the result of this theorem. The interested reader can restrict the proof in Besicovitch [3] to Borel sets and use the Heine-Borel theorem to give the result. □

The result of this theorem can be used to study the local structure of fractal sets, touching on *geometric measure theory*, but we instead focus on its consequences in *potential theory*. To do so, we introduce Egoroff's Theorem before giving two corollaries.

Theorem 3.10 *(**Egoroff's Theorem**, Jerrard [11, Theorem 4]). Let $D \subset \mathbb{R}^n$ be a Borel set and μ a measure with $\mu(D) < \infty$. Let $f_1, f_2, \ldots$ and f be functions from D to $\mathbb{R}$ such that $f_k(x) \to f(x)$ for each $x \in D$. Then for any $\delta > 0$, there exists a Borel subset $E \subset D$ such that*

$$\mu(D \backslash E) < \delta$$

and

$$\sup_{x \in E} |f_k(x) - f(x)| \to 0$$

as $k \to \infty$ (i.e. the sequence $\{f_k\}$ converges uniformly to f on E).

Proof. For $i, j \geq 1$, let

$$E_{i,j} := \{x \in D : \ |f_k(x) - f(x)| < 1/i \text{ for all } k \geq j\}.$$

For every i,

$$\lim_{j \to \infty} \mu(D \backslash E_{i,j}) = 0$$

by assumption, so we can find some $J(i)$ dependent on i such that

$$\mu(D \backslash E_{i,J(i)}) < \delta 2^{-i}.$$

Define

$$E = \bigcap_{i=1}^{\infty} E_{i,J(i)},$$

so the definition of $E_{i,j}$ implies that $|f_j - f| < 1/i$ on E for $j \geq J(i)$, thus $\{f_k\}$ converges uniformly to f on E. Now

$$\begin{aligned}
\mu(D \backslash E) &= \mu\left(D \backslash \bigcap_{i=1}^{\infty} E_{i,J(i)}\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} (D \backslash E_{i,J(i)})\right) && \text{by De Morgan's law} \\
&\leq \sum_{i=1}^{\infty} \mu(D \backslash E_{i,J(i)}) && \text{by (M3)} \\
&< \sum_{i=1}^{\infty} \delta 2^{-i} = \delta
\end{aligned}$$

by using the geometric sum $\sum_{i=1}^{\infty} 2^{-i} = 1$. □

Corollary 3.11 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ is a Borel set satisfying $0 < \mathcal{H}^s(F) < \infty$. There exists a compact set $E \subset F$ with $\mathcal{H}^s(E) > 0$ and a constant b such that*

$$\mathcal{H}^s(E \cap B(x, r)) \leq b r^s \tag{3.6}$$

for all $x \in \mathbb{R}^n$ and all $r > 0$.

Proof. Let $\mu = \mathcal{H}^s|_F$, the restriction of $\mathcal{H}^s$ to F. We see that μ is a mass distribution with $\mu(A) = \mathcal{H}^s(F \cap A)$ for a set A. For

$$F_1 := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} > 2^{1+s} \right\},$$

it follows that

$$\mathcal{H}^s(F_1) \leq 2^s 2^{-(1+s)} \mu(F) = \frac{1}{2} \mathcal{H}^s(F)$$

by Proposition 3.8 (ii). Thus

$$\mathcal{H}^s(F \backslash F_1) \geq \frac{1}{2} \mathcal{H}^s(F) > 0,$$

so if $E_1 = F \backslash F_1$, then $\mathcal{H}^s(E_1) > 0$ and

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} \leq 2^{1+s}$$

for $x \in E_1$. By Egoroff's Theorem, it follows that there exists a compact set $E \subset E_1$ with $\mathcal{H}^s(E) > 0$ and a number $r_0 > 0$ such that

$$\frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} \leq 2^{2+s}$$

for all $x \in E$ and all $0 < r \leq r_0$. However, we have that

$$\frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} \leq \frac{\mathcal{H}^s(F)}{r_0^s}$$

if $r \geq r_0$, so (3.6) holds for all $r > 0$. □

Corollary 3.12 *(Falconer [8]). Suppose $F \subset \mathbb{R}^n$ is a Borel set with $\mathcal{H}^s(F) = \infty$. There exists a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$, and such that for some constant b,*

$$\mathcal{H}^s(E \cap B(x,r)) \leq br^s$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$.

Proof. Theorem 3.9 provides us with a subset of F of positive finite measure, and applying Corollary 3.11 to this gives the result. □

3.4 Potential Theory

The mass distribution principle is used to estimate the mass of a large number of small sets, but introducing concepts of potential and energy, we can instead consider the local concentration of a mass. Then, by checking the convergence of a certain integral, we can find a lower bound of $\mathcal{H}^s$.

Definition 3.13 *(Falconer [8]).* Suppose that $F \subset \mathbb{R}^n$. Let μ be a mass distribution on $\mathbb{R}^n$ and let $s \geq 0$. The *s-potential* at a point $x \in \mathbb{R}^n$ with respect to μ is

$$\phi_s(x) := \int_F \frac{\mathrm{d}\mu(y)}{|x-y|^s}.$$

The *s-energy* of μ is given by

$$I_s(\mu) := \int_F \phi_s(x)\, \mathrm{d}\mu(x) = \iint_F \frac{\mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{|x-y|^s}.$$

We can connect these concepts to the Hausdorff dimension through mass distributions. Our aim is to show that mass distributions with finite s-energy are spread in such a way that each point has a concentration of mass sufficiently small, as to overcome the singularity of the integrand.

Theorem 3.14 *(Falconer [8, Theorem 3.13(a)]). Suppose that $F \subset \mathbb{R}^n$. If there exists a mass distribution μ on F such that $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_H F \geq s$.*

Proof. Suppose that μ is a mass distribution with support contained in F such that $I_s(\mu) < \infty$. Let

$$F_1 := \left\{ x \in F : \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} > 0 \right\}$$

be a subset of F. We first show that F_1 has zero mass by contradiction.
For $x \in F_1$, by the definition of $\limsup$, we can find a decreasing sequence of $\{r_i\}$ tending to 0 such that $\mu(B(x,r_i)) \geq \varepsilon r_i^s$ for some $\varepsilon > 0$. By Proposition 2.13,

$$\begin{aligned} I_s(\mu) &= \iint_F \frac{\mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{|x-y|^s} \\ &\geq \iint_{F_1} \frac{\mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{|x-y|^s} \\ &\geq \int_{|x-x_0|\leq\delta} \int_{|y-y_0|\leq\delta} \frac{\mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{|x-y|^s} \quad \to \quad \frac{\mu(x_0)\mu(y_0)}{|x_0-y_0|^s} \end{aligned}$$

as $\delta \to 0$. When $x_0 = y_0$, we find that $I_s(\mu) < \infty$ only when $\mu(\{x_0\}) = 0$, thus $\mu(\{x\}) = 0$ for all $x \in F_1$.

We can now find a decreasing sequence $\{q_i\}$, with $0 < q_i < r_i$ for all i, such that the annulus $A_i := B(x, r_i) \backslash B(x, q_i)$ has the following property:

$$\mu(A_i) \geq \frac{1}{4} \varepsilon r_i^s. \tag{3.7}$$

Taking subsequences if necessary, we may assume that the annuli A_i centred at x are pairwise disjoint with $r_{i+1} < q_i$. Hence, by Proposition 2.14, for all $x \in F_1$,

$$\begin{aligned}
\phi_s(x) &= \int_F \frac{\mathrm{d}\mu(y)}{|x-y|^s} \\
&\geq \sum_{i=1}^{\infty} \int_{A_i} \frac{\mathrm{d}\mu(y)}{|x-y|^s} \\
&\geq \sum_{i=1}^{\infty} \frac{1}{4} \varepsilon r_i^s r_i^{-s} = \infty
\end{aligned}$$

by (3.7) and as $|x-y|^{-s} \geq r_i^{-s}$ on A_i. Hence if $\mu(F_1) > 0$, then $\phi_s(x)$ is infinite over F_1 with respect to μ, thus we conclude that $\mu(F_1) = 0$.
So we consider $x \in F \backslash F_1$ such that

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} = 0 < c$$

for all $c > 0$. By Proposition 3.8 (i),

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(F \backslash F_1) \geq \frac{\mu(F \backslash F_1)}{c} \geq \frac{\mu(F) - \mu(F_1)}{c} = \frac{\mu(F)}{c}$$

since $\mu(F_1) = 0$. Hence $\mathcal{H}^s(F) = \infty$ and $\dim_H F \geq s$. □

This shows that if a mass distribution on a subset of $\mathbb{R}^n$ has finite s-energy, the Hausdorff dimension of this set has a lower bound of s. Now we consider a Borel set F to which we already have a dimensional lower bound, say $\dim_H F \geq s$. We show that there exists a mass distribution on F such that its t-energy is finite for values of t less than this bound s.

Theorem 3.15 *(Falconer [8, Theorem 3.13(b)]). Suppose $F \subset \mathbb{R}^n$ is a Borel set with $\mathcal{H}^s(F) > 0$. Then there exists a mass distribution μ on F such that $I_t(\mu) < \infty$ for all $t < s$.*

Proof. Suppose that F is a Borel set with $\mathcal{H}^s(F) > 0$. By Corollary 3.12, there exists a compact set $E \subset F$ with $0 < \mathcal{H}^s(E) < \infty$, and a constant b such that

$$\mathcal{H}^s(E \cap B(x,r)) \leq b r^s$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Let $\mu = \mathcal{H}^s|_E$, the restriction of $\mathcal{H}^s$ to E. We fix $x \in \mathbb{R}^n$ and let

$$m(r) := \mu(B(x,r)) = \mathcal{H}^s(E \cap B(x,r)) \leq br^s. \tag{3.8}$$

If $0 < t < s$, then

$$\begin{aligned} \phi_t(x) &= \int_{|x-y|\leq 1} \frac{\mathrm{d}\mu(y)}{|x-y|^t} + \int_{|x-y|>1} \frac{\mathrm{d}\mu(y)}{|x-y|^t} \\ &\leq \int_0^1 r^{-t}\,\mathrm{d}m(r) + \mu(\mathbb{R}^n) \end{aligned}$$

as $|x-y|^{-t}\mathbb{1}_{|x-y|<1}$ is bounded. Using integration by parts,

$$\begin{aligned} \phi_t(x) &\leq \left[r^{-t}m(r)\right]_{0^+}^1 + t\int_0^1 r^{-(t+1)}m(r)\,\mathrm{d}r + \mu(\mathbb{R}^n) \\ &\leq b + bt\int_0^1 r^{s-t-1}\,\mathrm{d}r + \mu(\mathbb{R}^n) && \text{by (3.8)} \\ &= b\left(1 + \frac{t}{s-t}\right) + \mathcal{H}^s(F). \end{aligned}$$

Thus for all $x \in \mathbb{R}^n$,

$$\phi_t(x) \leq c$$

for some constant c, so

$$\begin{aligned} I_t(\mu) &= \int_F \phi_t(x)\,\mathrm{d}\mu(x) \\ &\leq c\mu(\mathbb{R}^n) < \infty. \end{aligned}$$

□

4 Projections

The projection of a set can be thought of as its shadow onto another set, with real-life shadows being the projection of an object in 3-dimensional space onto a 2-dimensional plane. The dimension of a set is usually greater than that of its projection; in this section we explore projections of fractal sets, through which we can construct a set where this property does not always hold. We discuss how dimension changes through these mappings starting with the projection of 2-dimensional sets onto a line, before considering projections in higher dimensions.

To begin we must discuss what it means for a set to be projected onto a line. Let L_θ be the line in $\mathbb{R}^2$ passing through the origin at an angle θ with the horizontal axis. We let $\mathrm{proj}_\theta : \mathbb{R}^2 \to \mathbb{R}$ correspond to the orthogonal projection onto L_θ (see

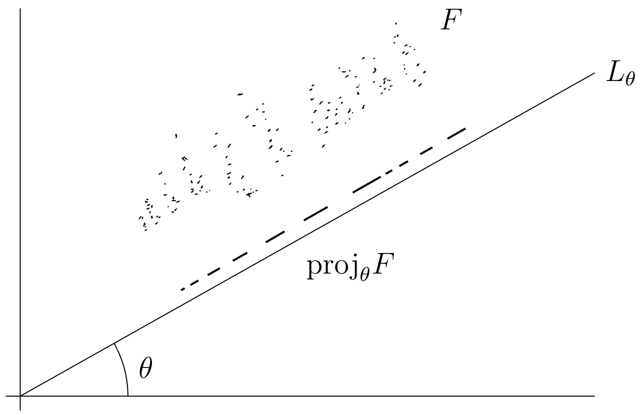

Figure 6: The projection of a set F onto the line L_θ

Figure 6). More precisely, we let $\boldsymbol{x} = (x, y)$ and $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$, a unit vector in the direction θ, such that

$$\mathrm{proj}_\theta : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto \boldsymbol{x} \cdot \boldsymbol{\theta}$$

where $\boldsymbol{x} \cdot \boldsymbol{\theta} = (x\cos\theta, y\sin\theta)$ is the usual scalar product. Clearly proj_θ is a Lipschitz mapping, i.e.

$$|\mathrm{proj}_\theta x - \mathrm{proj}_\theta y| \leq |x - y|$$

for all $x, y \in \mathbb{R}^2$, and as $\mathrm{proj}_\theta F \subset L_\theta$,

$$\dim_H(\mathrm{proj}_\theta F) \leq \min\{\dim_H F, 1\}$$

by Corollary 2.25. For certain fractal sets, such as the skew Cantor dust below, we use these properties to easily find a lower bound of the Hausdorff measure.

4.1 The Skew Cantor Dust

A fractal set often seen is the *Cantor dust* D, constructed as the Cartesian product $D = C \times C$, where C is the middle-thirds Cantor set. However, here we introduce the slight variant which we call the *skew Cantor dust* F.

We construct the set F by starting with E_0, the unit square in $\mathbb{R}^2$. We split this square into 16 smaller squares, each with side length $\frac{1}{4}$. Discarding all but four of these smaller squares, we are left with the set E_1 (note the arrangement of these squares in Figure 7). For each of these squares we repeat the process, so E_2 consists of 16 squares of side length $\frac{1}{8}$. We say that the limit of this process is the skew Cantor dust F, defined as the intersection of every E_k;

$$F = \bigcap_{k=0}^{\infty} E_k.$$

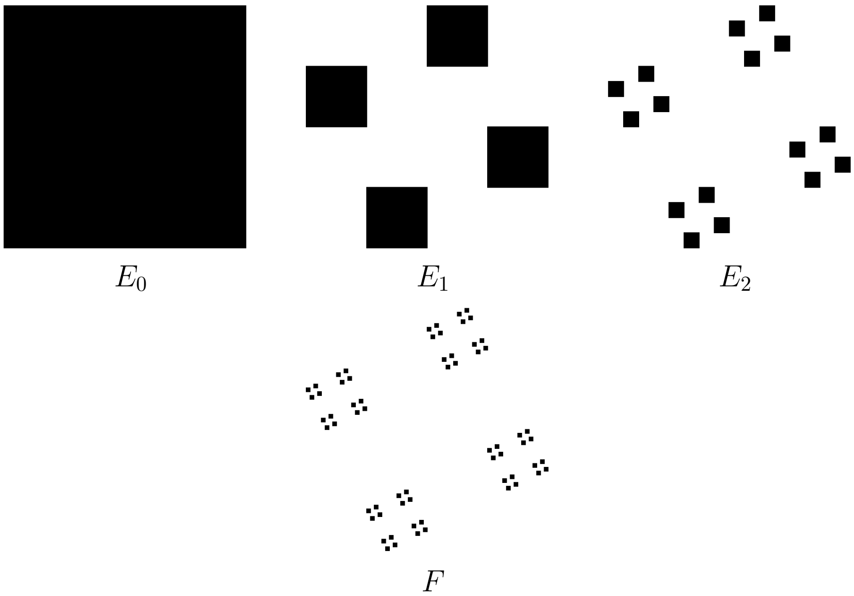

Figure 7: The skew Cantor Dust

We note that each E_k is the union of 4^k squares with side length 4^{-k}, so each level-k square has a diameter of $4^{-k}\sqrt{2}$ (the length across the diagonal).
As with the previous examples, we aim to find a non-zero positive upper and lower bound of $\mathcal{H}^s(F)$ for a specific s. The upper bound we find in much the same way as before, but we shall find the lower bound using a projection.

Upper bound

Considering the natural cover of F consisting of 4^k squares with side length 4^{-k}, we find a $\delta = 4^{-k}\sqrt{2}$-cover of F with

$$\mathcal{H}^s_\delta(F) = \inf_{\{U_i\}\in\mathcal{U}_\delta} \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\} \leq \sum_{i=1}^{4^k} (4^{-k}\sqrt{2})^s = 4^k(4^{-sk})\sqrt{2}^s.$$

Letting $k \to \infty$ gives $\delta \to 0$, so

$$\mathcal{H}^s(F) \leq \lim_{k\to\infty} 4^{k(s-1)}\sqrt{2}^s.$$

For $s = 1$,

$$\mathcal{H}^s(F) \leq \lim_{k\to\infty} 4^0 \cdot \sqrt{2}^1 = \sqrt{2},$$

an upper bound.

Lower bound

For the lower bound we consider proj_0, the orthogonal projection onto the horizontal axis. By the construction of F, we see that

$$\text{proj}_0 E_k = [0, 1]$$

for each k, thus

$$\text{proj}_0 F = [0, 1].$$

As proj_0 is a Lipschitz mapping, by Corollary 2.25,

$$\begin{aligned} \mathcal{H}^1(F) &\geq \mathcal{H}^1(\text{proj}_0 F) \\ &= \mathcal{H}^1([0, 1]) \\ &= \mathcal{L}[0, 1] = 1 \end{aligned}$$

by Proposition 2.19.

Thus when $s = 1$,

$$1 \leq \mathcal{H}^s(F) \leq \sqrt{2},$$

hence

$$\dim_H F = s = 1.$$

4.2 Dimensions of a Projected Set

We now turn our attention to the dimension of a projected set $F_\theta = \text{proj}_\theta F$, dependent on some angle $\theta \in [0, \pi)$. For some $s > 0$, if we can show that for some mass distribution μ_θ

$$\int_{F_\theta} I_s(\mu_\theta) \, \mathrm{d}\theta = \iiint_{F_\theta} \frac{\mathrm{d}\mu_\theta(x) \, \mathrm{d}\mu_\theta(y) \, \mathrm{d}\theta}{|x - y|^s} < \infty$$

for all θ, then $I_s(\mu_\theta) < \infty$ for almost all θ. Thus by Theorem 3.14,

$$\dim_H F_\theta \geq s$$

for almost all θ.

Remark. We say *almost all* θ as for some exceptional values of θ, we have that F_θ is a set of zero length.

Theorem 4.1 ***(Projection Theorem**, Falconer [8, Theorem 6.1]). Suppose that* $F \subset \mathbb{R}^2$ *is a Borel set.*

(i) If $\dim_H F \leq 1$, *then* $\dim_H F_\theta = \dim_H F$ *for almost all* $\theta \in [0, \pi)$.

(ii) *If* $\dim_H F > 1$*, then length* $(F_\theta) > 0$ *for almost all* $\theta \in [0, \pi)$ *and thus* $\dim_H F_\theta = 1$.

Proof.

(i): Suppose that $s < \dim_H F \leq 1$. By Theorem 3.15, there exists a mass distribution μ on a compact subset of F with $0 < \mu(F) < \infty$ and

$$\iint_F \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x-y|^s} < \infty. \tag{4.1}$$

For each θ, we define μ_θ on F_θ to be the mass distribution μ projected onto the line L_θ. Thus for each interval $[a, b] \subset \mathbb{R}$,

$$\mu_\theta([a,b]) = \mu\{x \in F:\ a \leq x \cdot \boldsymbol{\theta} \leq b\},$$

or equivalently

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}\mu_\theta(t) = \int_F f(x \cdot \boldsymbol{\theta})\,\mathrm{d}\mu(x)$$

for each non-negative function f. Now we aim to show that

$$\iint_F \frac{\mathrm{d}\mu_\theta(u)\,\mathrm{d}\mu_\theta(v)}{|u-v|^s} < \infty,$$

thus the result follows from Theorem 3.14.
Integrating over $[0, \pi)$,

$$\begin{aligned}
\int_0^\pi \left[\iint_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\theta(u)\,\mathrm{d}\mu_\theta(v)}{|u-v|^s}\right]\mathrm{d}\theta &= \int_0^\pi \left[\iint_F \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x\cdot\boldsymbol{\theta} - y\cdot\boldsymbol{\theta}|^s}\right]\mathrm{d}\theta \\
&= \int_0^\pi \left[\iint_F \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|(x-y)\cdot\boldsymbol{\theta}|^s}\right]\mathrm{d}\theta \\
&= \int_0^\pi \frac{\mathrm{d}\theta}{|\boldsymbol{\tau}\cdot\boldsymbol{\theta}|^s} \iint_F \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x-y|^s} \qquad (4.2)
\end{aligned}$$

for any fixed unit vector $\boldsymbol{\tau}$. This follows as the integral of $|(x-y)\cdot\boldsymbol{\theta}|^{-s}$ with respect to θ depends only on $|x-y|$, so we can separate the integral with the use of a fixed $\boldsymbol{\tau}$.
We have that the second part of (4.2) is finite by (4.1), thus it remains to show that

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\boldsymbol{\tau}\cdot\boldsymbol{\theta}|^s} < \infty.$$

We know

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\boldsymbol{\tau}\cdot\boldsymbol{\theta}|^s} = \int_0^\pi \frac{\mathrm{d}\theta}{|cos(\tau-\theta)|^s}$$

for which $|\cos(\tau-\theta)|^s$ is bounded below by Cx^s for some $C > 0$, thus as $s < 1$,

$$\int_0^\pi \frac{\mathrm{d}\theta}{|cos(\tau-\theta)|^s} < \infty.$$

Hence

$$\iint_F \frac{\mathrm{d}\mu_\theta(u)\,\mathrm{d}\mu_\theta(v)}{|u-v|^s} < \infty$$

for almost all $\theta \in [0, \pi)$. By the existence of such a μ_θ on F_θ and Theorem 3.14, $\dim_H F_\theta > s$, thus the result

$$\dim_H F_\theta = \dim_H F$$

follows.

(ii): Omitted, but follows a similar argument to (i) using *Fourier transforms* to show that projections have positive length. □

We may now expand this theorem to higher dimensions.

Theorem 4.2 *(Falconer [8, Theorem 6.2]). Let $G_{n,k}$ be the set of k-dimensional subspaces in $\mathbb{R}^n$, let proj_Π be the orthogonal projection onto the k-plane Π, and suppose $F \subset \mathbb{R}^n$ is a Borel set.*

(i) If $\dim_H F \leq k$, then

$$\dim_H(\mathrm{proj}_\Pi F) = \dim_H F$$

for almost all $\Pi \in G_{n,k}$.

(ii) If $\dim_H F > k$, then $\mathrm{proj}_\Pi F$ has positive k-dimensional measure, so

$$\dim_H(\mathrm{proj}_\Pi F) = k$$

for almost all $\Pi \in G_{n,k}$.

Proof. Omitted, but can be extended from the Projection theorem. □

From this, we see that if $F \subset \mathbb{R}^3$, then the projection $\mathrm{proj}_\Pi F$ onto a plane has

$$\dim_H(\mathrm{proj}_\Pi F) = \min\{\dim_H F, 2\},$$

so we can find the dimension of a set just by considering a view from a random direction. If the dimension from this view is less than 2, then we may assume this value to be equal to the dimension of the whole set.

We could calculate the dimension of an example set in this way, but we instead

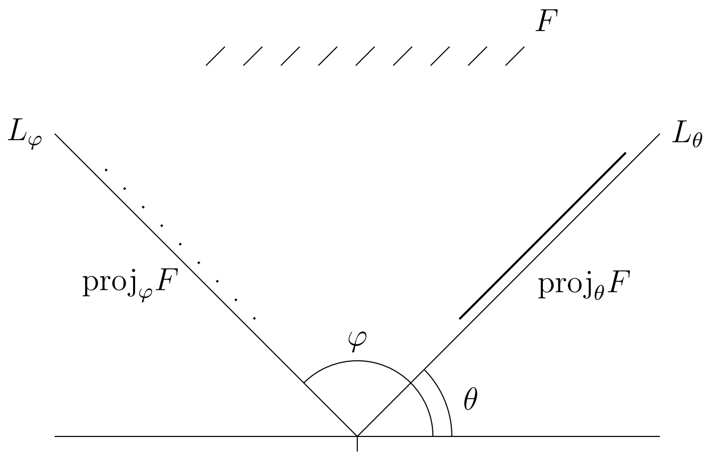

Figure 8: A Venetian blind set F projected onto the lines L_θ and L_φ

end this section by constructing sets with specific properties. For example, we can easily construct a set in $\mathbb{R}^2$ whose projection has positive length for one value of θ, but zero length for another. In fact, we can show that there exists a set whose projection is any set we desire, differing only by zero length.

Theorem 4.3 *(Falconer [8, Theorem 6.9]). Suppose that G_θ is a subset of L_θ for each $\theta \in [0, \pi)$. Then there exists a Borel set $F \subset \mathbb{R}^2$ such that*

(i) $\text{proj}_\theta F \supset G_\theta$ *for all θ, and*

(ii) length $(\text{proj}_\theta F \backslash G_\theta) = 0$ *for almost all θ.*

In particular, for almost all θ, the set of points of L_θ belonging to either G_θ or $\text{proj}_\theta F$*, but not both, have zero length.*

Proof. Omitted, but these sets use an 'iterated Venetian blind' construction (see Figure 8). By rotating a set through small angles, we achieve different length projections depending on θ. □

Again, this may be extended to higher dimensions; there exists a set $F \subset \mathbb{R}^n$ such that almost all projections of F onto k-dimensional subspaces differ from a set of our choosing by zero k-dimensional measure.
From this, we could hypothetically construct a set $F \subset \mathbb{R}^2$ such that the projection due to the position of the Sun in 3-dimensional space displays the time digitally onto the ground (see Figure 9). The displayed digits of this *digital sundial* would depend on the angle of projection θ, and would change throughout the day as the Sun moved across the sky. Due to the complexity of such a set, this would not be possible in reality, but theoretically this set does exist.

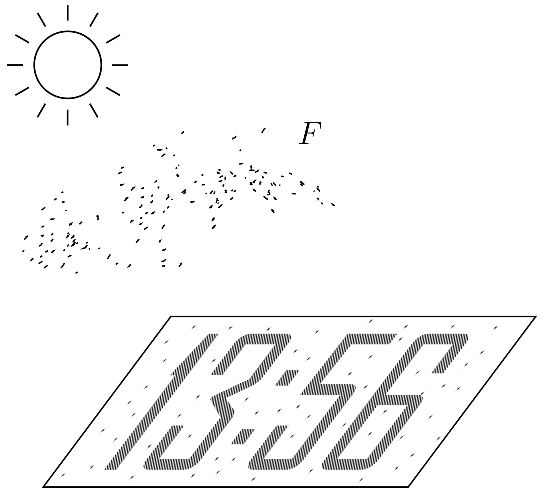

Figure 9: The set F projecting the time digitally onto the ground

5 Applications

This dissertation concludes with applications of the fractal dimension in areas beyond abstract mathematics. Many natural objects can be modelled as fractal sets, an example being the coastline of Britain as discussed in Section 2. We can analyse and compare the shape of irregular objects by using their fractal dimension. The Hausdorff dimension is often difficult to calculate, even with computer assistance, and so the examples below use variations of either the box-counting or similarity dimension, although the results are coherent with the Hausdorff dimension.

In neuroscience, many different cells of the brain exhibit statistical self-similarity and can be modelled as a fractal set. In response to trauma or neurodegeneration, astrocytes (cells that regulate the transmission of electrical impulses within the brain) change their structure and swell. Comparing the fractal dimension of these cells, we see differing values in a healthy patient as compared to a patient with a neurodegenerative disease, such as Alzheimer's disease (Pirici *et al.* [19] uses the box-counting dimension to analyse these structures). The cortical ribbon (a strip of grey matter surrounding the brain) is also affected by neurodegenerative diseases, with different fractal dimensions being observed in computer models of a healthy brain compared to one affected by Alzheimer's disease (King *et al.* [13] uses a variation of the box-counting dimension to study this).

Similarly, airways in the lungs can be modelled as a fractal set. Air from the trachea passes into the two lungs through the bronchi, which in-turn splits into many bronchioles; smaller alveoli branch off the bronchioles. These airways exhibit self-similarity, as with the astrocytes above, and so the fractal dimension can be calculated and analysed (Uahabi & Atounti [23] discusses how the box-counting dimension can be found). Patients with emphysema suffer from damaged alveoli that rupture due to weakness, creating larger air cavities. Using nuclear medicine scans, namely Single Photon Emission Computed Tomography (SPECT) scans, the fractal dimension of emphysematous lungs can be calculated; they can be shown to have a greater value than that of healthy lungs (Nagao *et al.* [18] uses a variation of the similarity dimension to study this).

References

[1] Robert Gardner Bartle. *The Elements of Integration*. John Wiley, 1966.

[2] Alexander C. R. Belton. MATH412: Topology and Fractals, 2019. The University of Lancaster.

[3] A.S. Besicovitch. On Existence of Subsets of Finite Measure of Sets of Infinite Measure. *Indagationes Mathematicae (Proceedings)*, 55:339–344, 1952.

[4] Michael Dickson. Non-Relativistic Quantum Mechanics. *Philosophy of Physics*, pages 275–415, 2007.

[5] Gerald Edgar. *Integral, Probability, and Fractal Measures*. Springer-Verlag New York, Inc., 1998.

[6] Gerald Edgar. *Measure, Topology, and Fractal Geometry*. Springer Science+Business Media, LLC, 2008.

[7] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Inc., 1992.

[8] Kenneth Falconer. *Fractal Geometry: Mathematical Foundations and its Applications*. John Wiley & Sons, 1990.

[9] Jan E. Grabowski. MATH112: Discrete Mathematics, 2016. The University of Lancaster.

[10] Michael Hochman. Lectures on Fractal Geometry and Dynamics, 2012. The Hebrew University of Jerusalem.

[11] Robert L. Jerrard. MAT1501: Geometric Measure Theory, Week 1, 2013. The University of Toronto.

[12] Mark Jerrum. MTH6126: Metric Spaces, 2010. Queen Mary University of London.

[13] Richard D. King, Brandon Brown, Michael Hwang, Tina Jeon, and Anuh T. George. Fractal Dimension Analysis of the Cortical Ribbon in Mild Alzheimer's Disease. *NeuroImage*, 53(2):471–479, 2010.

[14] Derek Kitson. MATH210: Real Analysis, 2017. The University of Lancaster.

[15] Seymour Lipschutz. *Theory and Problems of General Topology*. Schaum's Outline Series. McGraw-Hill, Inc., 1965.

[16] Benoit B. Mandelbrot. How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension. *Science*, 156(3775):636–638, 1967.

[17] Mathonline, Learn Mathematics. The Union and Intersection of Collections of Closed Sets. `http://mathonline.wikidot.com/the-union-and-intersection-of-collections-of-closed-sets`. Accessed 03/03/2020.

[18] Michinobu Nagao, Kenya Murase, Yoshifumi Yasuhara, and Junpei Ikezoe. Quantitative Analysis of Pulmonary Emphysema: Three-Dimensional Fractal Analysis of Single-Photon Emission Computed Tomography Images Obtained with a Carbon Particle Radioaerosol. *American Journal of Roentgenology*, 171(6):1657–1663, 1998.

[19] D. Pirici, L. Mogoantă, Otilia Mărgăritescu, Ionica Pirici, Valerica Tudorică, and Marieta Coconu. Fractal Analysis of Astrocytes in Stroke and Dementia. *Romanian Journal of Morphology and Embryology*, 50(3):381–390, 2009.

[20] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, Inc., third edition, 1976.

[21] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Book Co., 1987.

[22] Ambar N. Sengupta. Math 7312: Real Analysis, 2002. Louisiana State University.

[23] K. Lamrini Uahabi and M. Atounti. New Approach to the Calculation of Fractal Dimension of the Lungs. *Annals of the University of Craiova, Mathematics and Computer Science Series*, 44(1):78–86, 2017.

A Proof of Series Convergence

We aim to prove the following proposition, needed for the lower bound of the middle-thirds Cantor set (Section 2.4).

Proposition A.1. *Suppose that $X_i > 0$ for all i and $\sum_{i=1}^{\infty} X_i^s$ is convergent. Then for $\varepsilon > 0$ and $s \geq 0$,*

$$\sum_{i=1}^{\infty} \left(X_i + \frac{2\varepsilon}{2^i} \right)^s \to \sum_{i=1}^{\infty} X_i^s$$

as $\varepsilon \to 0$.

First we introduce the binomial series expansion.

Theorem A.2 *(**Binomial series**, Kitson [14, Proposition 5.29]). For $0 \leq x < 1$ and $\alpha \in \mathbb{R}$,*

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots.$$

Proof. Omitted. □

Sketch proof of Proposition A.1. For ε sufficiently small,

$$\begin{aligned}
\left(X_i + \frac{2\varepsilon}{2^i} \right)^s &= X_i^s \left(1 + \frac{2\varepsilon}{2^i} X_i^{-1} \right)^s \\
&= X_i^s + s\left(\frac{2\varepsilon}{2^i} X_i^{s-1} \right) + \frac{s(s-1)}{2!} \left(\frac{(2\varepsilon)^2}{2^{2i}} X_i^{s-2} \right) + \dots \\
&= X_i^s + s\left(2^{1-i} X_i^{s-1} \varepsilon \right) + s(s-1)\left(2^{1-2i} X_i^{s-2} \varepsilon^2 \right) + \dots
\end{aligned}$$

by Theorem A.2. Thus,

$$\begin{aligned}
\sum_{i=1}^{\infty} \left(X_i + \frac{2\varepsilon}{2^i} \right)^s &= \sum_{i=1}^{\infty} X_i^s + \sum_{i=1}^{\infty} s\left(2^{1-i} X_i^{s-1} \varepsilon \right) + \sum_{i=1}^{\infty} s(s-1)\left(2^{1-2i} X_i^{s-2} \varepsilon^2 \right) + \dots \\
&\to \sum_{i=1}^{\infty} X_i^s
\end{aligned}$$

as $\varepsilon \to 0$. □