The Hausdorff Dimension

Fractal Sets and Fractal Dimensions

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Preliminaries

Definition (Diameter)

The *diameter* of non-empty $U \subset \mathbb{R}^n$ is

$$|U| := \sup\{|x - y| : \ x, y \in U\}.$$

Definition (Cover)

A countable/finite collection of non-empty subsets $\{U_i\}$ *in* $\mathbb{R}^n$ *is a cover of a set* $F \subset \mathbb{R}^n$ *if*

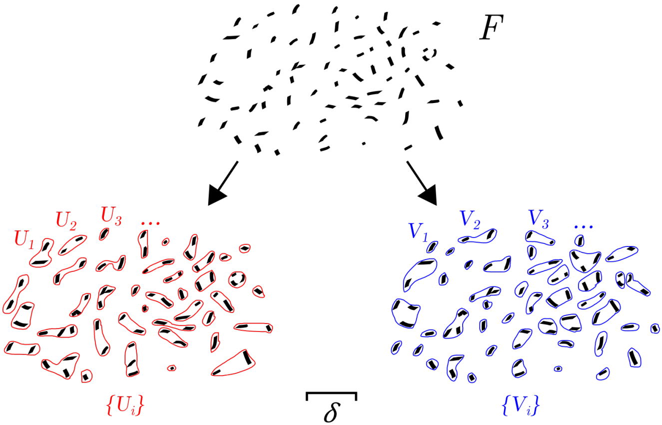
$$F \subset \bigcup_{i=1}^{\infty} U_i.$$

This is a δ*-cover if, for a given* $\delta > 0$*,*

$$|U_i| \leq \delta$$

for all i*.*

Preliminaries

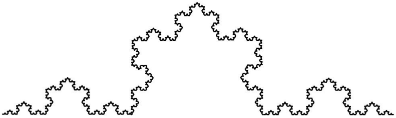


Two different δ-covers of a set F

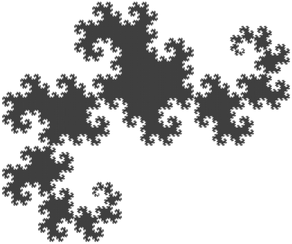
What is a fractal?

We say F is a fractal if the following are true:

(i) F has fine structure
(ii) F cannot be described with traditional geometry
(iii) Often F has some sort of self-similarity
(iv) Usually, the 'fractal dimension' of F is greater than its topological dimension
(v) F is often defined in a simple way



Von Koch curve



Heighway dragon

Hausdorff measure

Definition (s-dimensional Hausdorff content)

Suppose $F \subset \mathbb{R}^n$ and $s \geq 0$. We define the *s-dimensional Hausdorff content* as

$$\mathcal{H}^s_\delta(F) := \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

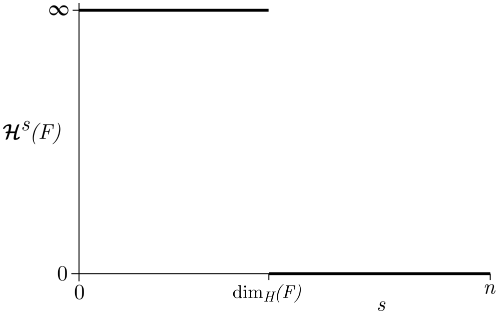
Definition (s-dimensional Hausdorff measure)

For $F \subset \mathbb{R}^n$, the s-dimensional Hausdorff measure is

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

Remark. We note that $\mathcal{H}^s$ gives a value on the extended real line $\mathbb{R} \cup \{\infty\}$.

Hausdorff measure



$\mathcal{H}^s(F)$ against s for a set F

Proposition

Let $F \in \mathbb{R}^n$, and let $t > s$. If $\mathcal{H}^s(F) < \infty$ then $\mathcal{H}^t(F) = 0$.

Hausdorff measure

Proposition

Let $F \in \mathbb{R}^n$, and let $t > s$. If $\mathcal{H}^s(F) < \infty$ then $\mathcal{H}^t(F) = 0$.

Proof. Consider $\{U_i\}$, a δ-cover of F. Then

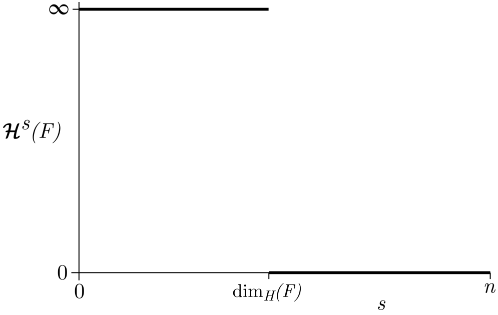
$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$

as $|U_i| \leq \delta$. Taking infima,

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$

Letting $\delta \to 0$, if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ for $t > s$. □

Hausdorff dimension



$\mathcal{H}^s(F)$ against s for a set F

Definition (Hausdorff dimension)

For a set $F \subset \mathbb{R}^n$, we define the Hausdorff dimension;

$$\dim_H(F) := \sup\{s : \mathcal{H}^s(F) = \infty\} = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\}.$$

Finding the dimension

Middle-thirds Cantor set F

We aim to find an upper bound of $\mathcal{H}^s(F)$ for a specific s.

Each E_k can be covered by 2^k intervals of length $\frac{1}{3^k}$. Letting $\delta = 3^{-k}$, it follows that

$$\mathcal{H}^s_{3^{-k}}(F) \leq \sum_{i=1}^{2^k} |U_i|^s = 2^k \left(\frac{1}{3^k}\right)^s = \left(2 \cdot 3^{-s}\right)^k.$$

Letting $s = \frac{\log 2}{\log 3}$,

$$\mathcal{H}^s_{3^{-k}}(F) \leq \left(2 \cdot 3^{-\frac{\log 2}{\log 3}}\right)^k = \left(2 \cdot \frac{1}{2}\right)^k = 1^k.$$

Thus when $k \to \infty$, we have $\delta = 3^{-k} \to 0$, so

$$\mathcal{H}^s(F) \leq 1,$$

a non-zero upper bound.

To conclude

Definition (s-dimensional Hausdorff content)

Suppose $F \subset \mathbb{R}^n$ and $s \geq 0$.

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Definition (s-dimensional Hausdorff measure)

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$$\dim_H(F) := \sup\{s : \mathcal{H}^s(F) = \infty\}$$